臺灣大學數學系

105 學年度下學期博士班資格考試題

科目：實分析　　　　　　　　　　2017.02.23

I. (20 pts) Let $f:\mathbb{R}^2\to\mathbb{R}$ be a Lebesgue integrable function satisfying $f\geq 0$ almost everywhere.

Prove that

(A) $$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dxdy=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dydx$$ (10 pts)

Note: Provide a detailed argument but NOT call it as Fubini's Theorem

Give a counterexample of formula (A). (10 pts)

II. (20 pts) Let the functions f_α be defined by

$$f_\alpha(x)=\begin{cases} x^\alpha \cos\dfrac{1}{x}, & x>0,\\ 0, & x=0.\end{cases}$$

Find all values of $\alpha\geq 0$ such that

(a) f_α is of bounded variation on $[0,1]$ (10 pts)

(b) f_α is absolutely continuous on $[0,1]$ (10 pts)

Justify your answers.

III. (20 pts) Let $f:[0,1]\to\mathbb{R}$ be a Lipschitz continuous function with

$$|f(x)-f(y)|\leq M|x-y|$$

for all $x,y\in[0,1]$, where M is a positive constant independent of x,y.

Prove that there exists a sequence of continuously differentiable functions $f_n:[0,1]\to\mathbb{R}$, $n=1,2,\cdots$ such that

(i) $|f_n'(x)|\leq M$ for all $x\in[0,1]$;

(ii) $f_n(x)\to f(x)$ for all $x\in[0,1]$

IV. (20 pts) Given $1 \le p \le \infty$ and $f \in L^p([0,\infty))$. Prove or disprove that

$$\lim_{n\to\infty} \int_0^\infty f(x) e^{-nx} dx = 0$$

V. (20 pts) Let $p \in [1,\infty)$. Prove that the unit ball of $L^\infty[0,1]$ is weakly closed in $L^p[0,1]$