臺灣大學數學系

103 學年度下學期博士班資格考試題

科目：實分析　　2015.03.05

2015 Spring, Real Analysis

You have to write your calculations and reasonings clearly.

(1)

(a) (3%) State and prove the Young's inequality.

(b) (3 %)State and prove the Hölder inequality for measurable functions.

(c) (4%)Prove the triangle inequality for $L^p(\Omega)$, where $1 < p < \infty$ and Ω is a measurable subset of $\mathbb{R}^n$.

(d) (15 %) Assume that $1 < p < \infty$ and Ω is a measurable subset of $\mathbb{R}^n$. Show that $L^p(\Omega)$ is complete.

(2) Prove the following inequalities.

(a) (10 %) Let $1 \le p \le +\infty$, $f \in L^p(\mathbb{R}^n)$ and $g \in L^1(\mathbb{R}^n)$. Then $f * g \in L^p(\mathbb{R}^n)$ and

$$\|f * g\|_p \le \|f\|_p \|g\|_1.$$

(b) (10 %) Let p and q satisfy $1 \le p, q \le \infty$ and $1/p + 1/q \ge 1$, and let r be defined by $1/r = 1/p + 1/q - 1$. If $f \in L^p(\mathbb{R}^n)$ and $g \in L^q(\mathbb{R}^n)$, then $f * g \in L^r(\mathbb{R}^n)$ and

$$\|f * g\|_r \le \|f\|_p \|g\|_q.$$

(3) Given $K \in L^1(\mathbb{R}^n)$ and $\epsilon > 0$, let

$$K_\epsilon(x) = \epsilon^{-n} K(\frac{x}{\epsilon}).$$

Prove that

(i) (5 %) $\int_{\mathbb{R}^n} K_\epsilon = \int_{\mathbb{R}^n} K$.

(ii) (5 %)$\int_{|x|>\delta} |K_\epsilon| \to 0$ as $\epsilon \to 0$, for any fixed $\delta > 0$.

(iii) (10 %) Let $f_\epsilon = f * K_\epsilon$, where $K \in L^1(\mathbb{R}^n)$ and $\int_{\mathbb{R}^n} K = 1$. If $f \in L^p(\mathbb{R}^n)$, $1 \le p < \infty$, then

$$\|f_\epsilon - f\|_p \to 0 \quad \text{as } \epsilon \to 0.$$

(4) (15 %) State and prove the Lebesgue's point theorem for $f(x) \in L^1_{loc}(\mathbb{R}^n)$.

(5) (10 %) Suppose $f(x) \in L^1(\mathbb{R}^n)$ and $\int_{\mathbb{R}^n} f(x)\phi(x)dx = 0$ for all $\phi \in C_c^\infty(\mathbb{R}^n)$. Show that $f(x) = 0$ for almost every $x \in \mathbb{R}^n$.

(6) (10%) Let $f(x, \xi) \in C^0(\mathbb{R} \times [-\pi, \pi])$ be a real continuous function such that $f(x + 1, \xi) = f(x, \xi)$ for all $(x, \xi) \in \mathbb{R} \times [-\pi, \pi]$. Suppose

$$\int_{-\pi}^{\pi} e^{ix\xi} f(x, \xi) d\xi = 0$$

for all $x \in \mathbb{R}$. Show that $f(x, \xi) = 0$ for all $(x, \xi) \in \mathbb{R} \times [-\pi, \pi]$.