臺灣大學數學系

101 學年度下學期博士班資格考試題

科目：實分析 2013.02.21

Qualified Exam for Real Analysis

Feb. 2013

1. (20%)

Let f be integrable on (a,b). Prove that there exists a unique function g such that

$$\int_a^x g(t)\,dt = \left(\int_a^x f(t)\,dt\right)^2, \forall a < x < b.$$

2. (20%)

Suppose that f and g are measurable functions on $\mathbb{R}^d$.

Recall the definition of the convolution of f and g given by

$$(f * g)(x) = \int_{\mathbb{R}^d} f(x-y)\,g(y)\,dy.$$

Show that $f * g$ is integrable whenever f and g are integrable, and that

$$\|f * g\|_{L^1(\mathbb{R}^d)} \le \|f\|_{L^1(\mathbb{R}^d)} \|g\|_{L^1(\mathbb{R}^d)},$$

with equality if f and g are non-negative.

3. (20%)

Suppose $\{\varphi_k\}_{k=1}^{\infty}$ is an orthonormal basis for $L^2(\mathbb{R}^d)$. Prove that the collection $\{\varphi_{j,k}\}_{j,k\ge 1}$ with $\varphi_{j,k}(x,y) = \varphi_j(x)\varphi_k(y)$ is an orthonormal basis of $L^2(\mathbb{R}^d \times \mathbb{R}^d)$.

4. (20%)

For $p,q > 1$, let

$$I_{p,q} = \inf\left\{\|u\|_p^p - \|u\|_q^q : u \in L^p \cap L^q \cap L^2, \|u\|_2 = 1\right\},$$

where L^s is the function space defined as $L^s = L^s(\mathbb{R}^2)$ with the norm $\|u\|_s = \left(\int_{\mathbb{R}^2} |u(x)|^s \, dx\right)^{\frac{1}{s}}$ for $u \in L^s$ and $s > 1$. Answer the following questions.

(i) Suppose $p > q > 2$. Which one holds true:

(A) $I_{p,q}$ is finite, (B) $I_{p,q} = \infty$, (C) $I_{p,q} = -\infty$.

(ii) Suppose $q > p > 2$. Which one holds true:

(A) $I_{p,q}$ is finite, (B) $I_{p,q} = \infty$, (C) $I_{p,q} = -\infty$.

Justify your answers rigorously.

5. (20%)

i. Calculate the value

$$I = \inf\left\{\int_{\mathbb{R}^2}\left[\frac{u^2(x)}{2} - \sqrt{1+u^2(x)} + 1\right]dx : \int_{\mathbb{R}^2} u^2(x)\,dx = 1\right\}.$$

ii. Can the value I be achieved by a function i.e. there exists $u \in L^2(\mathbb{R}^2)$ with $\int_{\mathbb{R}^2} u^2(x)\,dx = 1$ such that

$$\int_{\mathbb{R}^2}\left[\frac{u^2(x)}{2} - \sqrt{1+u^2(x)} + 1\right]dx = I?$$

Prove or disprove your result.