臺灣大學數學系
101 學年度上學期博士班資格考試題
科目：實分析 2012.09.13

1. (40%) For each of the following statements : Prove or disprove it.

   (a) Let f and $\{f_n\}_{n=1}^{\infty}$ be real functions defined on $\mathbb{R}$ such that for any $p \geq 1$, $f \in L^p(\mathbb{R})$ and $\{f_n\}_{n=1}^{\infty} \subset L^p(\mathbb{R})$. Is it possible that

   $$\lim_{n\to\infty} \|f_n - f\|_{L^p} = 0 \qquad \text{for all } \ p \geq 1,$$

   but $f_n(x) \to f(x)$ for no x ?

   (b) If there exists a positive constant M such that $\|f\|_{L^p(\mathbb{R})} < M$ for any $p > 1$, then $f \in L^1(\mathbb{R})$.

   (c) A perfect set must be uncountable.

   (d) Suppose $F : \mathbb{R}^n \to \mathbb{R}^n$ is a Lipschitz transformation. If $A \subset \mathbb{R}^n$ is a measurable set, then $F(A)$ is also measurable.

2. (15%)Suppose $f \in L(\mathbb{R}^n)$ and f^* is the Hardy-Littlewood maximal function of f. Prove that there exists a constant c independent of f and α such that

   $$\left|\left\{x \in \mathbb{R}^n \middle| f^*(x) > \alpha\right\}\right| \leq \frac{c}{\alpha}\int_{\mathbb{R}^n} |f|, \qquad \alpha > 0.$$

3. (10%) Let F be a closed subset of $\mathbb{R}$ and let $\delta(x) = \delta(x, F)$ be the corresponding distance function. If $\lambda > 0$ and f is nonnegative and integrable over the complement of F, prove that the function

   $$\int_{\mathbb{R}} \frac{\delta^{\lambda}(y) f(y)}{|x-y|^{1+\lambda}} dy$$

   is integrable over F.

4. (15%)State and prove the Fubini's Theorem.

5. (10%) Let

   $$I(p(t)) = \int_0^1 \left\{(p'(t))^2 - p^4(t)\right\} dt$$

   be a functional defined for all $p(t) \in C_0^1([0,1])$. Show that

   (a) I is not bounded above.

   (b) I is not bounded below.

   (c) 0 is a local minimum of I.

6. (10%) For functions f and g defined on the interval $[a, b]$, prove that if f and g have a common discontinuous point, then the Riemann-Stieltjes integral $\int_a^b f dg$ does not exist.