國立臺灣大學數學系

九十六學年度上學期博士班資格考試題

科目：機率

2007.09

Problems 1-4, 15 points each; Problems 5,6, 20 points each

1. Assume SLLN. Let an iid sequence with a positive L^1 rv X as common distribution to represent the life-time of, say, a bulb, Let rv $N(t)$ denote the " renewal number" up to time t. State and prove the LLN for $N(t)$. Let $f(t)$ be a continuous function on unit interval [0,1]. State and prove the so-called Monte Carlo convergence.

2. Use ch.f. to discuss the convergence in distribution of iid sequence X_n with symmetric r.v. X as common distribution such that the decay (1) $P\{|X| > x\} \approx x^{-(2+\delta)}, \delta > 0$, and (2) $P\{|X| > x\} \approx x^{-1}$.

3. Let $(X_n, \mathfrak{F}_n)$ be a martingale, and each X_n is in $L^2(dP)$. 1. prove that $(X_n^2, \mathfrak{F}_n)$ is a submartingale. 2. decompose X_n^2 into a sum of a martingale M_n and an increasing process A_n with $A_0 = 0$ (this is called Doob's decomposition, try to start with defining A_n), and explain that A_n is "predictable".

4. Let $(X_n, \mathfrak{F}_n)$ be a martingale, and each X_n is in $L^2(dP)$. the difference is $\xi_{m,n} := X_n - X_m, m < n$. prove 1. $E[\xi_{m,n}^2|\mathfrak{F}_m] = E[X_n^2|\mathfrak{F}_m] - X_m^2$. 2. if $\sum_n E\xi_{n-1,n}^2 < \infty$, then the martingale convergence holds both a.s. and in mean square.

5. What means a probability distribution to be stationary for a MC? Let X be a finite-states irreducible aperiodic MC. Then prove that there is at most one stationary distribution(try to use basic limit theorem). What means a MC to be doubly stochastic ? In case it is, write down the stationary distribution.

6. Let $B_t, t \geq 0$, denote the standard Brownian Motion on the line. 1. For each $t > 0$, define $\Delta_{n,j} := B(\frac{jt}{2^n}) - B(\frac{(j-1)t}{2^n})$; prove that $\sum_{j=1}^{2^n} \Delta_{n,j}^2$ converges in L^2 as n tends to ∞. 2. For any θ, $\exp \theta B_t - \theta^2 t/2, t \geq 0$, is a martingale.