臺灣大學數學系112學年度第1學期博士班一般資格考試

科目：機率論

2023.09.06

1. (15% = 4+4+7) Let $X, X_1, X_2, \ldots$ be random variables defined on the same probability space $(\Omega, \Sigma, \mathbb{P})$.
   (a) State the definitions of convergence in probability and almost sure convergence.
   (b) Prove that if $X_n \to X$ almost surely then $X_n \to X$ in probability.
   (c) Show that $X_n \to X$ in probability if and only if $\mathbb{E}[\min\{1, |X_n - X|\}] \to 0$ as $n \to \infty$.

2. (20% = 10 + 10) Let $(X_n)_{n\geq 1}$ be a supermartingale relative to the filtration $(\mathcal{F}_n)_{n\geq 1}$ such that $X_n \geq -2023$ almost surely for all $n \in \mathbb{N}$.
   (a) Show that X_n converges almost surely to an integrable random variable X.
   (b) Is it true that $X_n = \mathbb{E}[X \mid \mathcal{F}_n]$ for all n?

3. (20% = 5 + 5 + 10) Let $(B_t)_{t\geq 0}$ be a standard Brownian motion. Define a filtration $(\mathcal{F}_t)_{t\geq 0}$ by
$$\mathcal{F}_t = \sigma(\{B_s : 0 \leq s \leq t\}),$$
the σ-algebra generated by the random variables B_s for $0 \leq s \leq t$. Define, for $s \geq 0$,
$$\mathcal{F}_s^+ = \bigcap_{t>s} \mathcal{F}_t.$$
   (a) Show that $(\mathcal{F}_t^+)_{t\geq 0}$ is a filtration and $\mathcal{F}_t^+ \supseteq \mathcal{F}_t$ for all $t \geq 0$.
   (b) Is $\mathcal{F}_t^+ = \mathcal{F}_t$?
   (c) Prove the Blumenthal 0-1 law: if $A \in \mathcal{F}_0^+$ then $\mathbb{P}(A) \in \{0, 1\}$.

4. (25% = 5+10+10) Let $(X_n)_{n\geq 1}$ be a Markov chain on $S = \{0, 1, 2, \ldots\}$ with transition matrix P. Recall that a nonzero measure μ on S is said to be invariant if μ satisfies $\mu(\{i\}) < \infty$ for all $i \in S$ and
$$\mu(\{j\}) = \sum_{i\in S} \mu(\{i\}) P_{i,j} \quad \text{for all } j \in S.$$
Now, let $(p_k)_{k=0}^{\infty}$ be a sequence of real numbers such that $p_0 = 1$, $0 < p_k < 1$ for all $k \geq 1$ and $\prod_{k=1}^{\infty} p_k > 0$. Suppose that for $k \geq 1$, the transition probabilities from the site k are given by
$$P_{k,m} = \begin{cases} p_k & \text{if } m = k+1, \\ 1 - p_k & \text{if } m = 0, \\ 0 & \text{otherwise.} \end{cases}$$
   (a) Is this Markov chain irreducible?
   (b) Determine whether this Markov chain is recurrent or transient.
   (c) Does this Markov chain have an invariant measure? If yes, find such a measure.

5. $(20\% = 8 + 7 + 5)$

(a) Let X and $(X_n)_{n\geq 1}$ be random variables such that X_n converges to X in distribution as $n \to \infty$, and let $r > 0$. Show that if $(|X_n|^r)_{n\geq 1}$ is uniformly integrable then $\mathbb{E}|X_n|^r \to \mathbb{E}|X|^r$ as $n \to \infty$.

(b) Let $(Y_n)_{n\geq 1}$ be a sequence of i.i.d. exponential random variables with parameter 1. (So Y_1 satisfies $\mathbb{P}(Y_1 > t) = e^{-t}$ for all $t \geq 0$.) Define

$$U_n = \frac{(\sum_{i=1}^n Y_i) - n}{\sqrt{n}}.$$

Show that

$$\lim_{n\to\infty} \mathbb{E}|U_n| = \sqrt{\frac{2}{\pi}}.$$

(c) Using (b), prove the Stirling formula

$$\lim_{n\to\infty} \frac{\left(\frac{n}{e}\right)^n \sqrt{2n\pi}}{n!} = 1.$$

Hint: You may use the fact that $\sum_{i=1}^n Y_i$ has density

$$\frac{1}{(n-1)!} x^{n-1} e^{-x} \mathbf{1}_{(0,\infty)}(x).$$

Write $\mathbb{E}|U_n|$ as an integral in terms of the above density, split the integral into two at $x = n$, and substitute $u = x/n$.