臺灣大學數學系

106 學年度下學期博士班資格考試題

科目：機率論 2018.03.02

1. (5%+15%)

a) State the first and the second Borel-Cantelli lemma.

b) Let X_1, X_2,... be i.i.d. with $P(X_i > x) = e^{-x}$, let $M_n = \max_{1\le m\le n} X_m$. Show that $\limsup_{n\to\infty} X_n/\log n = 1$ a.s. and $M_n/\log n \to 1$ a.s..

2. (15%) Show that $\rho(F,G) = \inf\{\epsilon : F(x-\epsilon) - \epsilon \le G(x) \le F(x+\epsilon) + \epsilon \;\; \forall x\}$ defines a metric on the space of distributions and $\rho(F_n, F) \to 0$ iff $F_n \Rightarrow F$.

3. (15%) Let X_1, X_2,... be i.i.d. with $EX_i = 0$ and $EX_i^2 = \sigma^2 \in (0,\infty)$. Then

$$\sum_{m=1}^{n} X_m \Bigg/ \left(\sum_{m=1}^{n} X_m^2\right)^{1/2} \Rightarrow \chi$$

4. (20%)Let $(p_i : i \ge 1)$ be a sequence of numbers satisfying $p_i = 1 - q_i \in (0,1)$. Let $(X_n)_{n\ge 0}$ be a Markov chain on $\{0,1,2,...\}$ with transition probabilities

$$p_{i,i+1} = p_i, \;\; p_{i,i-1} = q_i \;\; \forall i \ge 1,$$

and $p_{0,0} = 1$. What is the probability of ultimate absorption at 0, having start at i?

5. (5%+10%)

a) Let B_t be a one-dimensional Brownian motion starting at 0. $a > 0$ and let $T_a = \inf\{t : B_t = a\}$. State the reflection principle.

b) Compute the distribution of $L = \sup\{t \le 1 : B_t = 0\}$.

6. (15%) Suppose that X_n is an adapted integrable process with $EX_T = EX_0$ for every bounded stopping time T. Show that X_n is a martingale. (Hint: construct a special stopping time.)