臺灣大學數學系
105 學年度上學期博士班資格考試題
科目：機率論

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請依題號次序作答

1. (25%=10+5+10)

   (a) Let $\{X_n\}_{n=1}^{\infty}$ be a sequence of independent and identically distributed (i.i.d.) random variables, and $S_n = X_1 + \cdots + X_n$. Suppose that $t\mathbf{P}(|X_1| > t) \to 0$ as $t \to \infty$. Prove that there exists a sequence $\{\mu_n\}_{n=1}^{\infty}$ such that $\frac{S_n}{n} - \mu_n \to 0$ in probability.

   (b) Give a counterexample such that, as $t \to \infty$, $t\mathbf{P}(|X_1| > t)$ does not converge to 0, and the above weak law does not hold.

   (c) Let $\{X_n\}_{n=1}^{\infty}$ be a sequence of i.i.d. random variables such that $\mathbf{P}(X_1 = 2^k) = 2^{-k}, k = 1, 2, \ldots$ Find a sequence $a_n > 0$ such that $S_n/a_n \to 1$ in probability, and prove it.

2. (30%=10+10+10) We write $\Rightarrow$ for the convergence in distribution. Let $\{X_n\}_{n=1}^{\infty}$ be a sequence of random variables and c be a constant.

   (a) Show that $X_n \to c$ in probability if and only if $X_n \Rightarrow c$.

   (b) Show that $X_n \Rightarrow X_\infty$ if and only if for every bounded and uniformly continuous function f one has $\mathbf{E}[f(X_n)] \to \mathbf{E}[f(X_\infty)]$ as $n \to \infty$.

   (c) Apply the results in (a) and (b) to show that if $X_n \Rightarrow X_\infty$ and $Z_n - X_n \Rightarrow 0$ then $Z_n \Rightarrow X_\infty$.

3. (15%=10+5) Let Let $Y_1, Y_2, \ldots$ be i.i.d. random variables with $0 < Y_n, \mathbf{E}[Y_1] = \mu \leq \infty$. Let $S_n = Y_1 + \cdots + Y_n$ and $N_t = \sup\{k : S_k \leq t\}$.

   (a) Show that $\frac{N_t}{t} \to \frac{1}{\mu}$ almost surely (a.s.).

   (b) Suppose that $\mu < \infty$ and $\mathrm{var}(Y_1) = \sigma^2 < \infty$. State a central limit theorem about N_t. No proof is needed.

4. (30%=18+12) Let $B_t, t \geq 0$, be a one dimensional Brownian motion starting at 0.

   (a) Show that $B_t^2 - t$ is a martingale. Find constants u, v such that $B_t^4 - utB_t^2 + vt^2$ is a martingale. Prove that, for any constant θ, $\exp(\theta B_t - \frac{\theta^2 t}{2})$ is a martingale.

   (b) Let $a > 0$ and $T = \inf\{t : B_t \notin (-a, a)\}$. Evaluate $\mathbf{E}[T]$ and $\mathbf{E}[T^2]$. Carefully state each property of Brownian motion that you use. No proof is needed.