臺灣大學數學系
99 學年度下學期博士班資格考試題
科目：偏微分方程 2011.02.24

1. (25 pts)
   (a) Solve the equation
   $$xu_x + u_y - 3u = -2e^y, u(0,y) = e^y.$$
   (b) Let $a \in \mathbb{R}$. Find all a such that the equation
   $$yu_x - xu_y = (y + ax^2)e^y$$
   has a C^1 solution in $\mathbb{R}^2$.

2. (25 pts) Assume $f \in C^2(\mathbb{R})$ and $g \in C^1(\mathbb{R})$. Solve the initial value problem for the one dimensional wave equation
   $$\begin{aligned} u_{tt} - u_{xx} = 0 \quad & \text{in } \mathbb{R} \times (0, \infty), \\ u = f, u_t = g \quad & \text{on } \mathbb{R} \times \{t = 0\}. \end{aligned}$$

3. (25 pts) Let Ω be a smooth bounded domain in $\mathbb{R}^n$ and $u(x,t)$ be a smooth solution of
   $$(0.1) \quad \begin{cases} u_t(x,t) = \Delta u(x,t) - u(x,t) - 2\int_\Omega |x-y|^2 u(y,t)\, dy & \text{in } \Omega \times (0,\infty) \\ u(x,t) = g(x) \quad \text{on } \partial\Omega \times (0,\infty), \end{cases}$$
   where Δ is the Laplacian in x variables.
   (a) Define
   $$F(w) = \int_\Omega [\frac{1}{2}|\nabla w(x)|^2 + \frac{1}{2}w(x)^2]\, dx + \int_\Omega\int_\Omega |x-y|^2 w(x)w(y)\, dxdy.$$
   Show that $F(u)$ is a non-increasing function of t.
   (b) Prove that equation (0.1) has a time independent weak solution $v(x)$ which satisfies
   $$\begin{cases} 0 = \Delta v(x) - v(x) - 2\int_\Omega |x-y|^2 v(y)\, dy & \text{in } \Omega \\ v(x) = g(x) \quad \text{on } \partial\Omega. \end{cases}$$
   (Hint: Use $F(w)$ as an energy functional.)

4. (25 pts) Let
   $$K(x,t) = \frac{1}{(4\pi t)^{1/2}} e^{-\frac{x^2}{4t}}, x \in \mathbb{R}, t > 0,$$
   $f(x) \in C(\mathbb{R})$, $f(x+1) = f(x)$ and $\int_0^1 f(x)\, dx = 3$. Define
   $$v(x,t) = \int_{\mathbb{R}} K(x-y,t) f(y)\, dy.$$
   (a) Show that $\lim_{(x,t)\to(x_0,0^+)} v(x,t) = f(x_0)$ for each $x_0 \in \mathbb{R}$.
   (b) Show that $\lim_{t\to\infty} v(0,t) = 3$.