臺灣大學數學系
107 學年度下學期博士班資格考試題
科目：偏微分方程 2019.02.22

1. Let Ω be an open bounded domain with smooth boundary in $\mathbb{R}^n$. Assume $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$.
(i)(10%) Prove the minimum principle for superharmonic functions, i.e., if $\Delta u \leq 0$ in Ω, then

$$\inf_{\Omega} u = \inf_{\partial\Omega} u.$$

(ii)(20%) Denote $D(\Omega)$ the diameter of Ω, i.e., $D(\Omega) = \sup\{|x-y| : x, y \in \Omega\}$. Show that

$$|u(x)| \leq \max_{\partial D} |u(x)| + \frac{D^2(\Omega)}{2n} \sup_{\Omega} |\Delta u(x)|, \quad \forall\, x \in \Omega.$$

(Hint: Construct an auxiliary function containing the term $\max_{\partial D} |u(x)| + \frac{1}{2n}[D^2(\Omega) - |x - x_0|^2] \cdot \sup_{\Omega} |\Delta u(x)|$ and function $u(x)$ with $x_0 \in \partial\Omega$. Then apply (i).)
(iii)(10%) Let $q(x) \in C^0(\overline{\Omega})$. Show that if $\max_{\overline{\Omega}} |q(x)| < \frac{D^2(\Omega)}{2n}$, then the boundary value problem $\Delta u + q(x)u = 0$ in Ω and $u = 0$ on $\partial\Omega$ admits only trivial solution.

2.(20%) Consider the heat equation with initial condition:

$$\begin{cases} \partial_t u - \partial_x^2 u = 0, & (t,x) \in (0,\infty) \times (-\infty,\infty), \\ u(0,x) = 1,\ x < 0; 0,\ x \geq 0. \end{cases} \tag{1}$$

Find a similarity solution $u(t,x) = t^{\alpha} v(x^{\beta}/t^{\gamma})$ with appropriate α, β, γ and $v(\cdot)$ solving (1).

3. Consider the initial value problem for the wave equation:

$$\begin{cases} \partial_t^2 u - \lambda_1 \partial_{x_1}^2 u - \lambda_2 \partial_{x_2}^2 u - \lambda_3 \partial_{x_3}^2 u = 0, & (t,x) \in (0,\infty) \times \mathbb{R}^3, \\ u(0,x) = \varphi(x), \quad \partial_t u(0,x) = \psi(x), \end{cases}$$

where $\varphi, \psi \in C^{\infty}(\mathbb{R}^3)$. Assume that $\lambda_1 \geq \lambda_2 \geq \lambda_3 > 0$ are constants.
(a)(20%) Derive the representation formula of the solution $u(x,t)$.
(b)(20%) Assume that $\varphi = 0$ and $\operatorname{supp}(\psi) \subset B(0,R)$, the ball centered at 0 with radius $R > 0$. Determine the support of $u(t,x)$ for $t > 0$.