臺灣大學數學系
104 學年度下學期博士班資格考試題
科目：偏微分方程 2016.02.26

1.(20%) Find an explicit formula of the solution to the following initial value problem:

$$\begin{cases} u_t = \Delta u + a u_{x_1} + bu, & x \in \mathbb{R}^n, t > 0, \\ u(x,0) = g(x), & x \in \mathbb{R}^n, \end{cases}$$

where a, b are nonzero real constants and g is a bounded function. Show that if $b < 0$, then for any $x \in \mathbb{R}^n$, $u(x,t) \to 0$ as $t \to +\infty$. Can the convergence be uniformly in x?

2.(20%) Solve the initial value problem:

$$u_t + \frac{1}{2}(u^2)_x = 0, \ t > 0,$$

$$u(x,0) = \begin{cases} 1, & x \le 0, \\ 1-x, & 0 \le x \le 1, \\ 0, & 1 \le x. \end{cases}$$

3.(a)(10%) Is there any nontrivial harmonic function u in $\mathbb{R}^n$ ($n \ge 2$) with bounded $L^p(\mathbb{R}^n)$ norm, where $1 < p < \infty$? Why?

(b)(20%) Let $f \in C^2(\mathbb{R}^2)$ with compact support. Define the Newtonian potential in $\mathbb{R}^2$

$$u(x) = \frac{1}{2\pi} \int_{\mathbb{R}^2} \log|x-y| f(y) dy. \tag{1}$$

(i) Show that u solves the Poisson equation $\Delta u = f$ in $\mathbb{R}^2$ and it has the asymptotic behavior

$$u(x) = \frac{C}{2\pi} \log|x| + O(|x|^{-1}) \quad \text{as} \quad |x| \to \infty. \tag{2}$$

Also, find the constant C.

(ii) Show that (1) is the only solution solving the Poisson equation $\Delta u = f$ with the asymptotic behavior (2).

4.(15%) Let Ω be a bounded smooth domain in $\mathbb{R}^3$. Define χ_Ω the characteristic function of Ω. Show that $\nabla\chi_\Omega \in H^{-1}(\mathbb{R}^3;\mathbb{R}^3)$, the dual space of $H_0^1(\mathbb{R}^3;\mathbb{R}^3)$. Find the precise formula of $\nabla\chi_\Omega(\phi)$ (meaning that $\nabla\chi_\Omega$ acts on ϕ) for any $\phi \in H_0^1(\mathbb{R}^3;\mathbb{R}^3)$ when Ω is the unit ball centered at the origin.

5.(15%) Let $u(x,y)$ be harmonic in the square $(-1,1)\times(-1,1)$. Assume that $u(x,\pm 1) = \pm\sin(2\pi x)$ for $x \in [-1,1]$ and $u(\pm 1, y) = \pm\sin(2\pi y)$ for $y \in [-1,1]$. Please give upper bounds and lower bounds of $\partial u/\partial x$ at $(0,0)$ and $(-1/2,-1/2)$.