臺灣大學數學系
103 學年度下學期博士班資格考試題
科目：偏微分方程　　2015.03.06

博士班資格考 (PDE)　　Spring 2015

1. (20%) Solve the following equations:

   (i) $u_x - u_y + u_z = zu, \quad u(x,0,0) = e^x$.

   (ii) $u_x^2 + u_y^2 = e^{2y}, \quad u(0,y) = 0$.

2. (30%) Let $B = \{x \in \mathbb{R}^n : |x| < 1\}$. Assume $u \in C^2(\overline{B})$ and $\Delta u \le 0$ in B.

   (i) Show that $u(x) \ge \min_{\partial B} u$ for $x \in B$.

   (ii) Assume further that u is a function of $|x|$ and $\Delta u < 0$ in B. Find the location of $\max_{\overline{B}} u$.

   (iii) Assume $w \in C^4(\overline{B})$, $\Delta^2 w + w = 0$ in B and $w = \Delta w = 0$ on ∂B. Show that $w = 0$ in B.

3. (30%) Let $H = H(x,t)$ be the heat kernel for the circle i.e. the solution of heat equation with the following form:

$$H(x,t) = \sum_{n=-\infty}^{\infty} a_n(t) e^{2\pi i n x} \qquad \text{for } x \in [0,1], t > 0,$$

   and $a_n(0) = 1$ for $n \in \mathbb{Z}$, where $i = \sqrt{-1}$.

   (i) Find $a_n(t)$ for $n \in \mathbb{Z}$.

   (ii) Prove that $\int_{-\frac{1}{2}}^{\frac{1}{2}} x^2 |H(x,t)|^2 \, dx = O(\sqrt{t})$ as $t \to 0$.

   (iii) Does $\lim_{t \to 0} \frac{x}{t} H(x,t)$ exist for $0 < x < 1$? Prove or disprove your answer.

4. (20%) Let $u = u(x,t)$ be the solution of the following problem:

$$\begin{cases} u_{tt} = \Delta u + \Gamma \dfrac{u^3}{1+u^2} & \text{for } x \in \Omega, t > 0, \\ u(x,t) = 0 & \text{for } x \in \partial\Omega, t > 0, \\ u(x,0) = f(x), u_t(x,0) = 0 & \text{for } x \in \Omega, \end{cases}$$

where $\Gamma \in \mathbb{R}$ is a constant, $f \in H^1(\Omega)$ and Ω is a bounded smooth domain in $\mathbb{R}^n$. Prove that there exists a constant $\gamma > 0$ such that $\sup_{t>0} \|u\|_{H_x^1(\Omega)} < \infty$ for $\Gamma < \gamma$.

Is there any convergence of solution u as $t \to \infty$?
Prove or disprove your answer.