臺灣大學數學系

103 學年度上學期博士班資格考試題

科目：偏微分方程 2014.09.19

1. (25 pts)

(a) Solve

$$\begin{cases} u_x - u_y = e^u, \\ u(x,0) = f(x) \quad \text{for } x \in \mathbb{R}. \end{cases}$$

(b) Find the solution $u(x,t)$ of the equation

$$\begin{cases} u_t + (3u^2)_x = 0 \quad \text{in } \mathbb{R} \times (0,\infty), \\ u(x,0) = f(x) \quad \text{for } x \in \mathbb{R} \end{cases}$$

which satisfies the Rankine-Hugoniot condition, where

$$f(x) = \begin{cases} 1, & \text{if } x \leq 0 \\ 1-x, & \text{if } 0 < x \leq 1. \\ 0, & \text{if } 1 < x \end{cases}$$

2. (25 pts) Let u be a solution of

$$\begin{cases} \Delta u = f \text{ in } B(x^0,1) \\ u = g \text{ on } \partial B(x^0,1). \end{cases}$$

Prove that there exists a constant C, depending only on the dimension n, such that

$$\max_{B(x^0,1)} |u| \leq C\left(\max_{B(x^0,1)} |g| + \max_{B(x^0,1)} |f|\right).$$

3. (25 pts) Find the solution $u(x,y,t)$ of the wave equation

$$u_{tt} = u_{xx} + u_{yy}, \; u(x,y,0) = xy$$

which satisfies the form $u(x,y,t) = f(2t-x,y)$, where f is a smooth function of two variables.

4. (25 pts) Suppose $u(x,t)$ is a smooth solution of

$$\begin{aligned} u_t - \Delta u - u(1-u) &= 0 \text{ in } U \times (0,\infty) \\ u &= 0 \text{ on } \partial U \times [0,\infty), \\ u(x,0) &= g(x), \end{aligned}$$

where U is a bounded open set in $\mathbb{R}^N$ with smooth boundary ∂U.

(a) Show that $0 \leq u \leq 1$ if $0 \leq g \leq 1$.

(b) Show that $u(x,t)$ is radially symmetric in x if U is a ball and g is radially symmetric.