# 臺灣大學數學系
## 100 學年度上學期博士班資格考試題
## 科目：數值偏微分方程 2011.09.15

**Do the first two problems. Choose two from problems 3, 4, 5.**

1. (30%) Consider the Poisson equation with variable coefficient on the square

$$\nabla \cdot (a(\mathrm{x})\nabla u) = f \text{ in } \Omega = [0,1] \times [0,1],\ \phi = 0 \text{ on } \partial\Omega,$$

where $f$ is smooth. Suppose $a(\cdot)$ is positive and smooth.

   (a) Design a second-order finite difference method to solve this equation numerically.
   (b) Write a pseudo-code for your scheme.
   (c) How do you show the convergence? (a short description)
   (d) Provide an iterative method to solve the corresponding linear system.
   (e) Design a preconditioner to speed up the iterative solver you propose.

2. (30%) Consider the initial value problem of the linear advection equation in one dimension

$$u_t + au_x = 0, x \in \mathbb{R}, t > 0,$$
$$u(x,0) = u_0(x).$$

Assuming $a > 0$ is a constant.

   (a) What is the upwind scheme? What is the Lax-Friedrichs scheme?
   (b) Write pseudocodes for these two schemes.
   (c) What are conditions for the stability of the two scheme, show the stability result (von Neumann analysis) for one of these two schemes.
   (d) If the initial condition

$$u_0(x) = \begin{cases} u_\ell & \text{for } x < 0 \\ u_r & \text{for } x > 0 \end{cases}$$

   where $u_\ell$ and $u_r$ are two constant. Find the error in $L_1$ at time $t$ for this initial datum.
   (e) In practics, we can only compute in finite domain, say $[0,1]$. How do you impose proper boundary conditions? How to implement them numerically?

3. (20%) Consider the finite difference equation

$$U_j^{n+1} = \alpha U_{j-1}^n + \beta U_j^n + \gamma U_{j+1}^n, j, n \in \mathbb{Z}, n \geq 0.$$

Define $\|U^n\|_2^2 := \sum_j |U_j^n|^2$.

   (a) Show that $\|U^n\|_2$ is uniformly bounded (independent of $n$) if and only if $\alpha, \beta, \gamma \geq 0$ and $\alpha + \beta + \gamma = 1$
   (b) Under the same condition, is it ture that $|U^n|_\infty := \sup_j |U_j^n|$ is uniformly bounded?

4. (20%) Consider the ordinary differential equation in $\mathbb{R}^n$

$$\frac{dy}{dt} = f(t,y), t > 0$$
$$y(0) = y_0$$

   (a) State and prove the convergence of the backward Euler method.
   (b) State and derive the second- and fourth-order Runge-Kutta method.

5. (20%) Discuss a subject or a topic in numerical PDE that you are familar with and show your own viewpoint.