臺灣大學數學系
106 學年度上學期博士班資格考試題
科目：數值方法　　　　2017.09.15

Numerical Methods
Ph.D qualify Exam.

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The contents include (1) basic numerical methods (including numerical linear algebra) and numerical PDEs. Write as much as you can. There is no fixed scale. It is to see how much you know and how deep you can think.

1. Consider the Poisson equation on the square

$$\triangle u = f \text{ in } \Omega = [0,1] \times [0,1],\ u = 0 \text{ on } \partial\Omega,$$

where f is smooth.

   (a) Design a second-order finite difference method for solving this equation.

   (b) Prove the convergence theorem.

   (c) What are the FFT and multigrid methods for solving this resulting linear system, their computational complexity? Write a pseudocode.

2. Consider the hyperbolic equation

$$u_t + au_x = 0, \quad x \in \mathbb{R}, t > 0, a > 0 \text{ a constant.}$$

Let us discretize the space by $j\Delta x$, $j \in \mathbb{Z}$ and time by $n\Delta t$, $n \geq 0$. We approximate $u(j\Delta x, n\Delta t)$ by U_j^n.

   (a) Show that the scheme

$$U_j^{n+1} - U_j^n = \frac{\Delta t}{2\Delta x} a \left(U_{j+1}^n - U_{j-1}^n \right)$$

   is unstable, while the upwind scheme

$$U_j^{n+1} - U_j^n = \frac{\Delta t}{2\Delta x} a \left(U_j^n - U_{j-1}^n \right)$$

   is stable under certain condition. What is that condition? Prove the stability theorem.

   (b) State and prove the Lax equivalent theorem for this system.

   (c) What is the Lax-Friedrichs scheme? What is the Lax-Wendroff scheme? What are their stability criteria?

3. Consider a fixed point iteration

$$x_n = g(x_{n-1}),$$

where $g : \mathbb{R} \to \mathbb{R}$ is a smooth function. Suppose this fixed point method does converge to a fixed point x^*. The Steffensen algorithm is an acceleration method to find x^* which reads

$$\hat{x}_n = x_{n-2} - \frac{(x_{n-1} - x_{n-2})^2}{x_n - 2x_{n-1} + x_{n-2}}.$$

or

$$x_{n+1} = G(x_n)$$

where

$$G(x) = x - \frac{(g(x) - x)^2}{g(g(x)) - 2g(x) + x}.$$

(a) Show that the Steffensen algorithm $\{x_k\}$ converges quadratically.

(b) Can you extend this method to two dimensions?

4. Consider the eigenvalue problem

$$Ax = \lambda x$$

where A is an $n \times n$ real or complex valued matrix. Its eigenvalues are expressed as

$$|\lambda_1(A)| \geq |\lambda_2(A)| \geq \cdots .$$

(a) Describe the power method which is to find the eigenvalue with largest magnitude, write a pseudocode.

(b) If λ_1 is not simple, can this method work?

(c) What can you do if the eigenspace of λ_1 is two dimension?

(d) What will you do if the invariant subspace corresponding to λ_1 is two dimension but the eigenspace corresponding to λ_1 is one dimensional?