臺灣大學數學系113學年度第2學期博士班一般資格考試

科目：幾何與拓樸

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(1) （10分) Prove that if a regular surface in $\mathbf{R}^3$ contains a straight line, then the surface has non-positive Gauss curvature at all the points of this line.

(2) （10分) Does there exist an embedded, compact minimal surface in $\mathbf{R}^3$? Why?

(3) （10分) Prove that that $O(n) = \{A | A \in GL(n, R), A^T A = I_n\}$ is a regular submanifold of

$$GL(n, R) = \{A | A \text{ is a } n \times n \text{ matrix with } det(A) \neq 0\},$$

(4) Let ∇ be the Levi-Civita connection of a Riemannian manifold (M, g). Recall that $R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$.
Let $\nabla^2_{X,Y} Z = \nabla_X \nabla_Y Z - \nabla_{\nabla_X Y} Z$.

(a) （5分) Show that $\nabla^2_{X,Y} Z - \nabla^2_{Y,X} Z = R(X,Y)Z$.

(b) （15分) Suppose S is a $(0,4)$-tensor. Recall that the covariant derivative tensor $\nabla_X S$ is defined by

$$\begin{aligned}&(\nabla_X S)(U,V,W,Z)\\ =&X(S(U,V,W,Z)) - S(\nabla_X U, V, W, Z) - S(U, \nabla_X V, W, Z) - S(U, V, \nabla_X W, Z) - S(U, V, W, \nabla_X Z)\end{aligned}$$

We denote by $\nabla^2_{X,Y} S$ the second covariant derivative of S:

$$\nabla^2_{X,Y} S = \nabla_X \nabla_Y S - \nabla_{\nabla_X Y} S.$$

Prove that

$$\begin{aligned}&(\nabla^2_{X,Y} S)(U,V,W,Z) - (\nabla^2_{Y,X} S)(U,V,W,Z)\\ =&- S(R(X,Y)U,V,W,Z)) - S(U,R(X,Y)V,W,Z) - S(U,R(X,Y)W,Z) - S(U,V,W,R(X,Y)Z)\end{aligned}$$

(5) A manifold (M^{2n}, ω) is called a symplectic manifold if ω is a closed two-form ($d\omega = 0$) and ω is non-degenerate, namely ω^n is a nowhere vanishing $2n$-form.
(a) （10分) Can ω be an exact form if M is compact (without boundary)?

Justify your answer.

(b) （10分) When will a unit sphere $S^{2n} \subset \mathbf{R}^{2n+1}$ admits a symplectic structure? Justify your answer. You may use the fact that $H^p(S^{2n}) = \mathbf{R}$ for $p = 0$ and $p = 2n$ and $H^p(S^{2n}) = 0$ for $1 \leq p \leq 2n - 1$.

(6) （10分) Let $f : M \to N$ be a smooth map between smooth manifolds, X and Y be smooth vector fields on M and N, respectively, and suppose that $f_*(X) = Y$. Then prove that $f^*(L_Y\omega) = L_X(f^*(\omega))$ where ω is a 1-form on N. Here L denotes the Lie derivative.

(7) （10分)For any two smooth vector fields X,Y on a smooth manifold M, prove the formula $[L_X, i_Y] = i_{[X,Y]}$ where L_X denotes the Lie derivative and i_X is the contraction of vector field acting on differential forms.

(8) （10分) Let M be a closed manifold and X be a vector field on M. Denote the flow generated by X by $\psi_t : M \to M$ defined by $\frac{d}{dt}\psi_t(x) = X_{\psi_t(x)}$ for any $x \in X$.

Given a function f, prove that $f \circ \psi_1 - f \circ \psi_0 = \int_0^1 \psi_t^*(df)(X)dt$.