國立臺灣大學數學系
108 學年度第2 學期博士班資格考試題
科目：幾何與拓樸
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(1) (10 分+10 分)
Let $U$ be a connected open subset of $\mathbf{R}^2$. Let $f$ be a smooth function on $U$ and let the metric

$$ds^2 = e^{2f}(dx^2 + dy^2).$$

(a) Determine the scalar curvature $R$ of the metric .
(b) Use the resulting formula above to find the scalar curvature of the upper half plane with the metric

$$ds^2 = \frac{dx^2 + dy^2}{y^2}.$$

(2) (10 分+10 分) Let $M$ be a n-dimensional Riemannian manifold.
(a) Let $\{e_1, \cdots, e_n\}$ be a local orthonormal frame for the tangent bundle. Let $\nabla$ be the Levi-Civita connection. Determine constants $\{a, b, c\}$ so that

$$g(\nabla_{e_i} e_j, e_k) = ag([e_i, e_j], e_k) + bg([e_j, e_k], e_i) + cg([e_k, e_i], e_j).$$

(b) Show that there exists a local orthonormal frame field with $[e_i, e_j] = 0$ for all $i, j$ if and only if the curvature tensor vanishes identically.

(3) (15 分) A derivation of $C^\infty(\mathbf{R}^n)$ based at a point $P$ is a linear map $L$ from $C^\infty(\mathbf{R}^n)$ to $R$ satisfying the Leibnitz rule $L(fg) = f(P)L(g) + L(f)g(P)$. Let $L$ be a derivation of $C^\infty(\mathbf{R}^n)$ based at $P = 0$, and let $x = (x^1, \cdots, x^n)$ be the coordinate functions on $\mathbf{R}^n$. Show there exist real constants $a_1, \cdots, a_n$ so that $L(f) = a_1 \frac{\partial f}{\partial x^1}(0) + \cdots + a_n \frac{\partial f}{\partial x^n}(0)$.

(4) (5 分 + 10分) Consider $(\mathbf{R}^2, g)$ to be the Riemannian manifold, with metric given by $g = (e^{-x} + y^2 e^x)dx^2 + xye^{-\frac{x}{2}}dxdy + 10(x^4 + y^4 + 5)dy^2$.
(a) Argue that this is a Riemannian metric.
(b) Is this a complete manifold? Prove or give a reason why it would not be.

(5) (15 分) Suppose that a Riemmanian manifold has section curvatures of both $+1$ and 1 at a point $p$: Prove there exist a (2-dimensional) tangent plane at $p$ that has zero sectional curvature.

(6) (5分 + 10 分) Let $X = \{(x, y, z) \in \mathbf{R}^3 | x^3 + xyz + y^2 = 1\}$.
(a) Show that $X$ is a 2-manifold.
(b) Consider the map $\pi : X \mapsto \mathbf{R}^2$ taking $(x, y, z)$to $(x, y)$. Find all points of $X$ at which $\pi$ fails to be a local diffeomorphism.