臺灣大學數學系
107 學年度上學期博士班資格考試題
科目：幾何與拓樸

2018.09.14

(1) [15 分 +10 分] Let $\mathbb{H}$ be the upper half plane $\{(x, y) \in \mathbb{R}^2 : y > 0\}$. For any $\alpha \in \mathbb{R}$, define the metric

$$g_\alpha = \frac{1}{y^\alpha}(\mathrm{d}x^2 + \mathrm{d}y^2) \ .$$

(a) If $\alpha \neq 2$, prove that $(\mathbb{H}, g_\alpha)$ is *incomplete.*

(b) Write (x, y) as $z = x + iy$. For any $(a, b, c, d) \in \mathbb{R}^4$ with $ad - bc = 1$, show that

$$z \mapsto \frac{az + b}{cz + d}$$

defines an isometry of $(\mathbb{H}, g_2)$.

(2) [25 分] Let $\mathbb{H}$ be the upper half plane $\{(x, y) \in \mathbb{R}^2 : y > 0\}$, and S^1 be the circle $\{e^{i\theta}\}$. Consider the following metric on $\mathbb{H} \times S^1$

$$g = \frac{\mathrm{d}x^2 + \mathrm{d}y^2}{y^2} + \left(\mathrm{d}\theta + \frac{1}{y}\mathrm{d}x\right)^2 \ .$$

Denote $y\partial_x - \partial_\theta$ by e_1, $y\partial_y$ by e_2 and ∂_θ by e_3.

Calculate its curvature R_{2112}, R_{3113} and R_{3223}, where

$$R_{jiij} = \left\langle (\nabla_{e_i}\nabla_{e_j} - \nabla_{e_j}\nabla_{e_i} - \nabla_{[e_i, e_j]})e_i, e_j \right\rangle \ .$$

(3) [20 分] Let M be a *hyperbolic* manifold. Suppose that $\gamma_0 : S^1 \to M$ is a closed geodesic, whose γ_0' has constant length. Is it possible to find a one-parameter family of closed curves

$$\gamma : S^1 \times \{t \in \mathbb{R} : -\epsilon < t < \epsilon\} \to M$$

with

$$\gamma(\,\cdot\,, 0) = \gamma_0(\,\cdot\,) \quad \text{and} \quad \left.\frac{\partial \gamma}{\partial t}\right|_{t=0} \perp \gamma_0' \quad \text{everywhere on } \gamma_0 \ ,$$

such that

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} L[\gamma(\,\cdot\,, t)] < 0 \ ?$$

Give your reason.

Here, $L[\gamma(\cdot, t)]$ means the arc length of the closed curve $\gamma(\,\cdot\,, t) : S^1 \to M$.

(4) [5 分 +5 分 +10 分] On $\mathbb{R}^3$, consider the 1-form

$$a = \mathrm{d}z + \frac{1}{2}(x\,\mathrm{d}y - y\,\mathrm{d}x)\ .$$

(a) Calculate $\mathrm{d}a$ and $a \wedge \mathrm{d}a$.

(b) Note that $\ker a$ is everywhere 2-dimensional. Check that $\mathrm{d}a|_{\ker a}$ is everywhere non-degenerate.

(c) Suppose that U and V are vector fields defined on the unit ball B, which are pointwise linearly independent and belongs to the kernel of a. Prove that $[U, V]$ is nowhere in the kernel of a.

(5) [10 分] Let Σ be a genus 2 surface (closed and oriented). Suppose that $f : \Sigma \to \Sigma$ is a continuous map which is homotopic to the identity map. Show that f must admit a fixed point.