國立臺灣大學數學系
103學年度上學期博士班資格考試題
科目：幾何與拓撲

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1. Recall that a vector field J along a geodesic γ is called a Jacobi field if

$$\frac{D^2 J}{dt^2} + R(\gamma'(t), J(t))\gamma'(t) = 0.$$

(a) (15 points) If J is a Jacobi vector field along a geodesic γ , show that

$$g(J(t), \gamma'(t)) = tg(J(0), \gamma'(0))$$

for all t.

(b) (5 points) Prove that if $J(t_0) = 0$ for some t_0, then $J'(0)$ must be orthogonal to $\gamma'(0)$.

2. Consider the metric $g = A^2(r)dr \otimes dr + r^2 d\theta \otimes d\theta + r^2 \sin^2\theta d\phi \otimes d\phi$ on $M = I \times S^2$, where r is a local coordinate on $I \subset R$ and (θ, ϕ) are spherical local coordinates on S^2.

(a) (10 points) Compute the Ricci curvature and the scalar curvature of this metric.

(b) (5 points) What happens when $A(r) = \frac{1}{\sqrt{1-r^2}}$?

(c) (5 points) What happens when $A(r) = \frac{1}{\sqrt{1+r^2}}$?

(d) (5 points) For which functions $A(r)$ is the scalar curvature constant?

3. (15 points) Let $f : M \mapsto R$ be a proper, distance-nonincreasing function on a Riemannian manifold (M, g), so $|f(x) - f(y)| \leq dist_M(x, y)$. Here d_M is the distance function induced by the Riemannian metric g. Prove that (M, g) is complete. (Recall that "proper" means that the preimage of a compact set is compact.)

4. Let M be a n-dimensional Riemannian manifold with the Levi-Civita connection ∇. Given a smooth function $f \in C^\infty(M)$, we can define the Hessian of f (denoted by $Hess(f)$) as follows:

$$Hess(f)(X, Y) = X(Y(f)) - (\nabla_X Y)(f)$$

where X and Y are smooth vector fields.
(a) (5 points) Prove that $Hess(f)(X,Y) = Hess(f)(Y,X)$.
(b) (5 points) Prove that $Hess(f)(hX,Y) = Hess(f)(X,hY) = hHess(f)(X,Y)$ for any smooth function h.
(c) (10 points) One can define the Laplacian of f as the trace of the Hessian, i.e.

$$\triangle f = \sum_{i,j=1}^{n} g^{ij} Hess(f)(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}) = \sum_{i,j=1}^{n} g^{ij} \Big(\frac{\partial^2 f}{\partial x^i \partial x^j} - \sum_{k=1}^{n} \Gamma_{ij}^{k} \frac{\partial f}{\partial x^k} \Big).$$

Prove that $\triangle f = \frac{1}{\sqrt{|g|}} \sum_{i,j=1}^{n} \frac{\partial}{\partial x^i}(\sqrt{|g|} g^{ij} \frac{\partial f}{\partial x^j})$ where $|g| = det(g_{ij})$. (Hint: You may use the fact that the Christoffel symbols are $\Gamma_{ij}^{k} = \frac{1}{2} \sum_{l=1}^{n} g^{kl}(\frac{\partial g_{jl}}{\partial x^i} + \frac{\partial g_{il}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^l})$. First show that $\frac{\partial}{\partial x^k} \sqrt{|g|} = \frac{1}{2} \sqrt{|g|} \sum_{p,q=1}^{n} g^{pq} \frac{\partial g_{pq}}{\partial x^k}$.)

5. (20 points) Let M be a Riemann surface (i.e. a 2-dimensional Riemannian manifold) with metric g. Define a new metric $\overline{g} = e^f g$ for some smooth function f. If $R_{\overline{g}}$ and R_g are the scalar curvatures of the two metrics $\overline{g}$ and g respectively, show that

$$R_{\overline{g}} = e^{-f}(\triangle_g f + R_g)$$

where $\triangle_g$ denotes the Laplacian (on functions) in the g-metric.