臺灣大學數學系

98 學年度下學期博士班資格考試題

科目：代數　　　　　　2010.02.25

Do all the problems.

1. (20 pts) Classify groups of order 18 up to isomorphism.

2. (10 pts) Let Q_8 be the group of quaternions. That is,

   $Q_8 := \{\pm 1, \pm i, \pm j, \pm k | i^2 = j^2 = k^2 = -1, ij = -ji = k, jk = -kj = i, ki = -ik = j\}.$

   Determine the minimum n such that Q_8 can be embedded in S_n.

3. (20 pts) Let k be a field. In the polynomial ring $R = k[x, y, z, w]$, we consider the ideal $I = (xz - y^2, yw - z^2, xw - yz)$. Show that I is prime. Is R/I a UFD?

4. (10 pts) Let $\mathbb{Q}$ be the field of rational numbers and let $\overline{\mathbb{Q}}$ be an algebraic closure of $\mathbb{Q}$. Let E be a *maximal* subfield of $\overline{\mathbb{Q}}$ not containing $\sqrt[3]{2}$ (Such field exists by Zorn's lemma). Show that every finite extension of E is a cyclic extension.

5. (15 pts) Let $\mathbb{F}_q$ be the finite field of $q = p^n$ elements. Let $K = \mathbb{F}_q(X)$ be the rational function field. Consider the group G of automorphisms of K obtained by

   $$X \mapsto \frac{aX + b}{cX + d}$$

   with $a, b, c, d \in \mathbb{F}_q$ and $ad - bc \neq 0$. Prove the following statements:

   (a) The order of $G = q^3 - q$.

   (b) The fixed field of G is equal to $\mathbb{F}_q(Y)$, where

   $$Y = \frac{(X^{q^2} - X)^{q+1}}{(X^q - X)^{q^2+1}}.$$

6. (10pts) Let $\mathbb{C}$ be the field of complex numbers. Let

   $$A = \begin{bmatrix} 0 & 0 & 0 & 3 \\ 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \in M_4(\mathbb{C}).$$

   Let

   $$C_A(\mathbb{C}) = \{B \in M_4(\mathbb{C}) \mid BA = AB\}.$$

   Then $C_A(\mathbb{C})$ is a finite dimensional $\mathbb{C}$-vector space. Find $\dim_{\mathbb{C}} C_A(\mathbb{C})$.

7. (15 pts) Let $a, b \in \mathbb{Q}^\times$. Let B be the 4-dimensional $\mathbb{Q}$-vector space with a $\mathbb{Q}$-basis $\{1, \alpha, \beta, \gamma\}$. We endow B with a structure of $\mathbb{Q}$-algebra as follows: The identity is 1, and the multiplication table of $\{\alpha, \beta, \gamma\}$ is given by

$$\alpha^2 = a,\ \beta^2 = b,\ \alpha\beta = \gamma = -\beta\alpha.$$

Then B is a non-commutative $\mathbb{Q}$-algebra. Show that

$$\{(x, y, z) \in \mathbb{Q}^3 \mid ax^2 + by^2 = z^2\} \neq \{(0,0,0)\} \iff B \cong M_2(\mathbb{Q}).$$

Hint: ($\Rightarrow$) To construct an isomorphism $i : B \cong M_2(\mathbb{Q})$, use Cayley-Hamilton theorem to guess possible $i(\alpha)$ and $i(\beta)$ in $M_2(\mathbb{Q})$. ($\Leftarrow$) Suppose $i : B \cong M_2(\mathbb{Q})$. To prove the converse, first show that one may assume $i(\alpha) = \begin{bmatrix} 0 & a \\ 1 & 0 \end{bmatrix}$ by replacing i with $g^{-1}ig$ for a suitable $g \in GL_2(\mathbb{Q})$.