臺灣大學數學系113學年度第1學期博士班一般資格考試

科目：代數

2024.09.06

All rings and algebras (over a field k) are assumed to be commutative.

Problem 1. Prove or disprove the following statements.

(1) Let $f : R \to S$ be a ring homomorphism. If m is a maximal ideal of S, then $f^{-1}(m)$ is a maximal ideal of R.

(2) Let E/F and F/K be finite Galois field extensions. If both $\mathrm{Gal}(E/F)$ and $\mathrm{Gal}(F/K)$ are abelian, then E/K is also a Galois extension with abelian Galois group.

(3) $\mathrm{SO}(2, \mathbb{R})$ is normal in $\mathrm{SL}(2, \mathbb{R})$.

(4) There exists a representation $\rho : G \to \mathrm{GL}(n, \mathbb{C})$ of finite group such that $\mathrm{Tr}(\rho(g)) = \sqrt[3]{2}$ for some $g \in G$.

Problem 2. Let $n \in \mathbb{Z}_{>0}$ and let A be an $n \times n$ matrix with coefficients in $\mathbb{C}$. Show that

$$\det(e^A) = e^{\mathrm{Trace}(A)}.$$

Problem 3. Show that there is no finite simple group of order 72.

Problem 4. Let p and q be two prime numbers. Compute

$$\mathbb{Z}\left[\frac{1}{p}\right] \otimes_{\mathbb{Z}} \mathbb{Z}\left[\frac{1}{q}\right].$$

Problem 5. Let

$$z = \cos(2\pi/13) + \cos(10\pi/13).$$

Show that $\mathbb{Q}(z)/\mathbb{Q}$ is a Galois extension of degree 3.

Problem 6. Let k be a field. We admit the following statement:

Theorem (Zariski's lemma)**.** *Let K/k be a field extension. If K/k is finitely generated as a k-algebra, then* $[K : k]$ *is finite.*

(1) Let $f : A \to B$ be a k-algebra morphism of finitely generated k-algebras. Show that if m is a maximal ideal of B, then $f^{-1}(m)$ is a maximal ideal of A.

(2) Let R be a finitely generated k-algebra and let $f \in R$. Show that if f is contained in every maximal ideal of R, then f is nilpotent. (Possible hint: consider the localization map $R \to R_f$.)

(3) (Bonus: do only after you finish all the problems) Prove Zariski's lemma.

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