# 臺灣大學數學系112學年度第2學期博士班一般資格考試

## 科目：代數

2024.02.21

**1.** For a subset $X$ of $\mathbb{Z}$, let

$$R^{(n)}(X) \coloneqq \{ f(t) \in \mathbb{R}[t] | \, f(X) \subset \mathbb{Z}, \deg f \leq n\}$$

and $R(X) \coloneqq \bigcup_{n=0}^{\infty} R^{(n)}(X)$. By definition, $R^{(n)}(X)$ form $\mathbb{Z}$-modules and $R(X)$ forms a commutative ring.

(a) (**5 points.**) Show that $R^{(n)}(X)$ is a free $\mathbb{Z}$-module for $n \geq 0$ if $X$ is an infinite set.

(b) (**5 points.**) Find an explicit basis of $R^{(n)}(\mathbb{Z})$ for $n \geq 0$

(c) (**10 points.**) Is $R(\mathbb{Z})$ Noetherian? Explain.

(d) (**10 points.**) Show that $R(X) = R(\mathbb{Z})$ if and only if $X$ is dense in $\mathbb{Z}_p \coloneqq \varprojlim \mathbb{Z}/p^n\mathbb{Z}$ for all prime number $p$.

**2.** Let $\sigma$ be a field automorphism of $\mathbb{R}$; let $p$ be a prime number and $\tau$ a field automorphism of $\mathbb{Q}_p$ (the field of fractions of $\mathbb{Z}_p \coloneqq \varprojlim \mathbb{Z}/p^n\mathbb{Z}$).

(a) (**5 points.**) Prove that $\sigma$ preserves the order of $\mathbb{R}$, i.e., $\sigma(x) < \sigma(y)$ for $x < y$.

(b) (**5 points.**) Prove that $\sigma$ must be the identity map, i.e., $\sigma(x) = x$ for $x \in \mathbb{R}$.

(c) (**10 points.**) Prove that $\tau(\mathbb{Z}_p) = \mathbb{Z}_p$.

(d) (**5 points.**) Prove that $\tau$ must be the identity map, i.e., $\tau(x) = x$ for $x \in \mathbb{Q}_p$.

**3.** Let $S, U$ denote the images of the matrices

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$$

in

$$\mathrm{PSL}_2(\mathbb{Z}) \coloneqq \mathrm{SL}_2(\mathbb{Z}) \Big/ \left\{ \pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\},$$

respectively.

(a) (**5 points.**) Prove that any element of $\mathrm{PSL}_2(\mathbb{Z})$ of finite order has order of either $1, 2, 3$.

(b) (**10 points.**) Classify all the finite subgroups of $\mathrm{PSL}_2(\mathbb{Z})$ up to isomorphism.

(c) (**10 points.**) Prove that $\mathrm{PSL}_2(\mathbb{Z})$ is generated by $S$ and $U$.

**4.** Let $P(t) \coloneqq t^4 - 30t^2 + 1$.

(a) (**5 points.**) Show that $P(t)$ is irreducible over $\mathbb{Q}$.

(b) (**10 points.**) Show that, despite (1), $(P(t) \bmod p) \in \mathbb{F}_p[t]$ is <u>reducible</u> for all prime number $p$.

(c) (**5 points.**) Let $K$ denote the splitting field of $P(t)$ over $\mathbb{Q}$. Find $\mathrm{Gal}(K/\mathbb{Q})$ and all the intermediate fields of the extension $K/\mathbb{Q}$.