臺灣大學數學系111學年度第1學期博士班一般資格考試

科目：代數

2022. 09. 12

1. **(20 points.)** Let S_n and A_n denote the symmetric group and the alternating groups on n letters, respectively. Here you may assume the fact that A_n is simple for $n \geq 5$.
   (a) Prove that if H is a simple subgroup of S_n of order > 2, then H is contained in A_n.
   (b) Prove that A_6 has no subgroup of index 4.
   (c) Let G be a group of order 90. Prove that either G has a normal subgroup of order 5 or there is a nontrivial homomorphism from G to S_6.
   (d) Prove that there are no simple groups of order 90.
2. **(20 points.)** Let D be an integral domain. A function $N : D \to \mathbb{Z}_{\geq 0}$ is said to be a Dedekind-Hasse norm on D if
   (i) $N(0) = 0$,
   (ii) $N(a) > 0$ if $a \neq 0$, and
   (iii) for any nonzero elements a and b in D, either $b|a$ or there exist elements x and y in D such that $0 < N(xa - yb) < N(b)$.

   Also, a nonzero element d of D is said to be a universal side divisor if d is not a unit and has the property that for any a in D, either $d|a$ or there exists a unit u in D such that $d|(a - u)$.
   (a) Prove that if an integral domain has a Dedekind-Hasse norm, then it is a principal ideal domain.
   (b) Prove that if an integral domain is a Euclidean domain, but not a field, then it has a universal side divisor.
3. **(15 points.)** Let F be an extension of $\mathbb{Q}$ of degree 4 that is not Galois over $\mathbb{Q}$. Prove that the Galois closure of F has Galois group either S_4, A_4, or D_8 (the dihedral group of order 8). Prove that the Galois group is dihedral if and only if F contains a quadratic extension of $\mathbb{Q}$.
4. **(15 points.)** Let R be an integral domain that is Noetherian and integrally closed. Let K be its field of fractions. Assume that I is a nonzero ideal of R and r is an element of K such that $rI \subseteq I$. Prove that $r \in R$.
5. **(15 points.)** Let R be a commutative Noetherian ring with 1. Prove that the ring $R[[x]]$ ($:= \{\sum_{n=0}^{\infty} a_n x^n : a_n \in R\}$) of formal power series over R is also Noetherian. (*Hint*: Given an ideal I of $R[[x]]$, consider $J_d := \{a \in R : ax^d + \cdots \in I\}$ for $d = 0, 1, 2, \ldots$.)
6. **(15 points.)** Let R be a ring. Let

$$0 \longrightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \longrightarrow 0$$

   be an exact sequence of R-modules. Prove that there exists an R-module homomorphism $\phi : B \to A$ such that $\phi \circ \alpha = \mathrm{id}_A$ if and only if there exists an R-module homomorphism $\psi : C \to B$ such that $\beta \circ \psi = \mathrm{id}_C$, where id_A and id_C denote the identity maps on A and C, respectively.