臺灣大學數學系
109 學年度上學期博士班資格考試題
科目：代數　　2020.09.18

Notations:
$\mathbb{N}$: the set of natural numbers.
$\mathbb{Z}$: the ring of integers.
$\mathbb{Q}$: the field of rational numbers.
$\mathbb{F}_n$: the finite field with n elements.
$GL_n(F)$: the group of non-singular $n \times n$ matrices over the field F.
$SL_n(F) := \{A \in GL_n(F) | \det(A) = 1\}$.

(1) (30%)
   (a) Determine all Sylow p-subgroups of $SL_2(\mathbb{F}_3)$ with p being a prime dividing $|SL_2(\mathbb{F}_3)|$. (Justify your answers)
   (b) Find generators for a Sylow p-subgroup of the symmetric group S_{2p}, where p is an odd prime. Show that this is an abelian group of order p^2.
   (c) Find generators for a Sylow p-subgroup of the symmetric group S_{p^2}, where p is a prime. Show that this is a non-abelian group of order p^{p+1}.

(2) (20%)
   (a) Show that if F is a field, then $F[[x]]$ is a PID whose only ideals are 0, $F[[x]]$ and $\langle x^k \rangle$ for $k \in \mathbb{N}$.
   (b) Show that there exists an irreducible polynomial of degree 5 in $\mathbb{Z}_{11}[x]$.

(3) (20%)
   (a) Let M be a finitely generated module over a PID. Show that if N is a submodule of M, then N and M/N are also finitely generated and $\operatorname{rank} M = \operatorname{rank} N + \operatorname{rank} M/N$.
   (b) Show that if M is a finitely generated R-module (R : commutative with 1) and $IM = M$ (I : an ideal of R contained in the Jacobson radical of R), then $M = 0$.

(4) (30%)
   (a) Is every finite group isomorphic to some Galois group $\mathrm{Gal}(F/K)$ for some extension F of some field K? Justify your answer.
   (b) Are the following polynomial equations solvable by radicals over $\mathbb{Q}$? Explain your answers.
      (i) $x^n - 1 = 0$, $n \geq 7$, $n \in \mathbf{N}$.
      (ii) $x^5 - 7x^2 + 7 = 0$.