臺灣大學數學系
100 學年度下學期博士班資格考試題
科目：代數 2012.02.24

(1) (20%) Let p be a prime number of the form $4k+3$.

(a) Prove that either

$$\left(\frac{p-1}{2}\right)! \equiv 1 (\text{mod } p) \quad \text{or} \quad \left(\frac{p-1}{2}\right)! \equiv -1 (\text{mod } p).$$

(b) The product of all the positive even integers less than p is congruent modulo p to either 1 or -1.

(2) (20%) Let H and K be normal subgroups of a finite group G. Suppose that $G/H \simeq K$.

(a) Give an example to show that G/K need not be isomorphic to H.

(b) Prove that if H is simple, then $G/K \simeq H$.

(3) (20%) Let $R = \mathbb{Z}[2x, 2x^2, 2x^3, \dots] \subset \mathbb{Z}[x]$.

(a) Find a prime ideal $\mathfrak{m}$ which is minimal over the ideal (2).

(b) Find a chain of prime ideals $(0) = \mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \cdots \subsetneq \mathfrak{p}_r = \mathfrak{m}$ such that the length r is maximal.

(4) (10%) Let R be a commutative ring, I be an ideal in R and M a finitely generated R-module. Suppose that $IM = M$, prove that $(1+a)M = 0$ for some $a \in I$.

(5) (20%)

(a) Find a Galois extension E over $\mathbb{Q}$ such that $\text{Gal}(E/\mathbb{Q})$ is cyclic of order 16.

(b) Find a Galois extension E over $\mathbb{Q}$ such that $\text{Gal}(E/\mathbb{Q})$ is cyclic of order 8.

(6) (10%) Let $V = \mathbb{R}[x]$ be the real vector space of polynomials. Define the inner product

$$\langle f, g \rangle = \int_0^1 fg\, dx.$$

Let D be the differentiation. Prove that the adjoint D^* of D (i.e. $\langle Df, g\rangle = \langle f, D^*g\rangle$) does not exist.

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