台灣大學數學系

九十三學年度博士班入學考試題

迴歸分析

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Suppose that we observe $Y_i = \beta_0 + \beta_1 X_{i1} + \cdots + \beta_k X_{ik} + \varepsilon_i$, $i = 1, \cdots, n$. Here $\beta = (\beta_0, \beta_1, \cdots, \beta_k)^T$ is a vector of constant parameters, $X = \left(x_1, \cdots, x_n\right)^T$, with $x_i = (1, X_{i1}, \cdots, X_{ik})^T$, is a constant design matrix, and $\varepsilon = (\varepsilon_1, \cdots, \varepsilon_n)^T$ is a vector of independent random variables with zero mean. Assume that X is of full rank. The least squares estimate of β, denoted as $\hat{\beta}$, is the solution to

$$\min_{\beta} \sum_{i=1}^{n} \left(Y_i - \beta_0 + \beta_1 X_{i1} + \cdots + \beta_k X_{ik}\right)^2 .$$

Let $\hat{Y} = (\hat{Y}_1, \cdots, \hat{Y}_n)^T = X\hat{\beta}$ and $e = (e_1, \cdots, e_n)^T = Y - \hat{Y}$.

(1)

Suppose that Var $(\varepsilon_i) = \sigma^2$, $i = 1, \cdots, n$.

(a)

(7 points) Show that $Y^T Y = \hat{Y}^T \hat{Y} + e^T e$.

(b)

(7 points) Show that Var $\left(\hat{Y}_i\right) \leq$ Var $\left(Y_i\right)$, $i = 1, \cdots, n$.

(c)

(6 points) Find a unbiased estimate of σ^2 based on e.

(2)

Suppose that $\varepsilon_i \sim$ Normal $\left(0, \sigma^2\right)$, $i = 1, \cdots, n$.

(a)

(10 points) Find the maximum likelihood estimates of β and σ^2 and denote them as $\tilde{\beta}$ and $\tilde{\sigma}^2$.

(b)

(10 points) Find the distributions of $\tilde{\beta}$ and $\tilde{\sigma}^2$.

(c)

(10 points) Find the likelihood ratio test of $H_0 : \sigma^2 = \sigma_0^2$ versus $H_1 : \sigma^2 \neq \sigma_0^2$, where σ_0^2 is a given positive constant.

(d)

(10 points) Find the likelihood ratio test of $H_0 : \beta_k = 0$ versus $H_1 : \beta_k \neq 0$.

(3)

Let $H = (h_{ij}) = X(X^T X)^{-1} X^T$. Let $\hat{\beta}_{(i)}$ be the least squares estimate of β obtained after deleting the ith observation. Let $\hat{Y}_i(i) = x_i^T \hat{\beta}_{(i)}$ where x_i^T is the ith row of X.

(a)

(7 points) Show that $0 \leq h_{ii} \leq 1$ for all $i = 1, \cdots, n$.

(b)

(7 points) Show that $\hat{\beta}_{(i)} = \hat{\beta} - (X^T X)^{-1} x_i e_i / (1 - h_{ii})$.

(c)

(6 points) Show that $\hat{Y}_i - \hat{Y}_i(i) = h_{ii} e_i / (1 - h_{ii})$.

(4)

Suppose that Var $(\varepsilon_i) = c_i^2 \sigma^2$, $i = 1, \cdots, n$. The weighted least squares estimate of β, denoted as $\hat{\beta}_{WLS}$, is the solution to

$$\min_{\beta} \sum_{i=1}^{n} c_i^{-2} \left(Y_i - \beta_0 + \beta_1 X_{i1} + \cdots + \beta_k X_{ik} \right)^2 .$$

(a)

(10 points) Show that both $\hat{\beta}$ and $\hat{\beta}_{WLS}$ are unbiased for β.

(b)

(10 points) Show that Var $(a^T \hat{\beta}_{WLS}) \leq$ Var $(a^T \hat{\beta})$ for any $(k+1)$-vector a.