國立臺灣大學數學系

九十七學年度博士班入學考試試題

科目：偏微分方程

2008.04.25

There are many problems here. Do at least three problems to show you have enough background. Also try to answer the question as deeper as possible to show your depth.

1. State and prove the weak and strong versions of the maximal principle for the Laplace equation on a bounded domain in $\mathbb{R}^n$. Give an application of the maximal principle.

2. State and prove the Poincaré inequality for bounded domain. Find an application?

3. Construct an entropy solution for the equation

$$u_t + \left(\frac{u^2}{2}\right)_x = 0.$$

with initial data

$$u(x,0) = \begin{cases} 1 & \text{for } |x| < 1 \\ 0 & \text{for } |x| \geq 1 \end{cases}$$

How about the same question for the equation: $u_t + (u^3/3)_x = 0$?

4. (a) Consider $\triangle u = 0$ in a ball $|x| < 1$ in $\mathbb{R}^n$ with boundary condition $\frac{\partial u}{\partial \nu} = g$ on $|x| = 1$. Is it true the solution always exists? (Prove if it exists, or give the counterexample, or find the condition to guarantee the existence.)

   (b) Consider the Poisson equation $\triangle u = f$ with Robin boundary condition $\partial u/\partial\nu + \alpha u = 0$ on a bounded domain Ω. Under what condition the solution is unique (a natural sufficient condition)? Prove your argument.

5. Construct the fundamental solution of wave equation in $\mathbb{R}^3$ and $\mathbb{R}^2$.

6. Consider $u_t + f(u)_x = u_{xx}$. Show that for any convex function $\eta(u) \geq 0$, the integral

$$\int \eta(u(x,t))\, dx$$

is non-increasing in time. Use this to show that $\|u(\cdot,t)\|_{L^p}$ is non-decreasing in time.