臺灣大學數學系

九十二學年度博士班入學考試題

偏微分方程

Do as many problems as you could.

A. (a) Please state the Holmgren theorem.
(b) Please state Fritz John's Global Holmgren theorem.
(c) Use (b) to find the maximum region in which u vanishes, where u solves the lateral Cauchy problem

$$\begin{cases} \partial_t^2 u - c^2 \partial_x^2 u = 0 & \text{in } \mathbb{R}_t\{x \geq 0\} \\ u(t,0) = u_x(t,0) = 0 & 0 \leq t \leq T. \end{cases}$$

Here c and T are positive constants.

B. Let Ω be an open bounded domain in $\mathbb{R}^n$ with smooth boundary. Use the Lax-Milgram theorem to show that there exists a unique weak solution $u \in H_0^1(\Omega)$ to

$$\sum_{i,j=1}^{n} \partial_{x_j}(a_{ij}(x)\partial_{x_i} u) = f$$

for $f \in (H_0^1(\Omega))^*$, the dual space of $H_0^1(\Omega)$, where $a_{ij}(x) \in L^\infty(\Omega)$ for all i, j and there exists a constant $\delta > 0$ such that for all $x \in \Omega$ and any vector $\xi = (\xi_1, \cdots, \xi_n)$

$$\sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \geq \delta|\xi|^2.$$

C. (a) Verify that $G_t(x) = (4\pi t)^{-n/2}\exp(-|x|^2/(4t))$ for $t > 0$ is the fundamental solution of the heat operator $L = \partial_t - \Delta$, i.e.,

$$\begin{cases} LG_t = 0 & \text{for} \quad x \in \mathbb{R}^n, t > 0 \\ \lim_{t\to 0^+} G_t(x) = \delta_0(x). \end{cases}$$

(b) Consider the initial value problem for the heat equation

$$\begin{cases} \partial_t u - \Delta u = 0 \quad \text{for} \quad x \in \mathbb{R}^n, t > 0 \\ u(x,0) = f(x). \end{cases}$$

Assume that $f(x) \in L^1(\mathbb{R}^n)$ and is non-negative. Then show that $u(x,t)$ is also non-negative for all $t > 0$ and

$$\|u(\cdot,t)\|_{L^1(\mathbb{R}^n)} = \|f\|_{L^1(\mathbb{R}^n)} \quad \forall\, t > 0.$$

D. Let $u(x,t)$ be the solution of

$$\begin{cases} \partial_t^2 u - \Delta u = 0 \quad \text{in} \mathbb{R}^3 \times \mathbb{R} \\ u(x,0) = 0,\ u_t(x,0) = f(x) \quad \forall\, x \in \mathbb{R}^3, \end{cases}$$

where $f(x)$ is a non-negative smooth function with compact support. Show that if $u(x_0,t) = 0$ for some x_0 and for all t, then $u \equiv 0$.

E. Find $u(x,t)$ solving

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0 \quad \text{in } 0 < x < \infty,\ t \in \mathbb{R} \\ u_t(0,t) + a u_x(0,t) = 0 \\ u(x,0) = 0, \quad u_t(x,0) = V \end{cases}$$

where V, a, and c are positive constants with $a > c$.