臺灣大學數學系

九十學年度博士班入學考試題

偏微分方程式

There are problems A. to F. You have to do 4 out of the 6 problems.

A.

Let $\Omega \subset R^2$ be a domain and Γ be a C^1 curve inside Ω such that $\Omega - \Gamma$ consists of two disjoint subdomains Ω_1 and Ω_2. A function $g \in C(\Omega - \Gamma)$ is said to have a jump $[g](P)$ at a point $P \in \Gamma$ if both limits in the following formula exists

$$[g](P) = {}_{Q\in\Omega_1}Pg(Q) - {}_{Q\in\Omega_2}Pg(Q).$$

Consider a linear equation $a(x,y)u_x + b(x,y)u_y = c(x,y)$ where a, b and c are C^1 functions defined on Ω. Assume that $u(x,y)$ defined on Ω satisfies either (a) $u \in C^1(\Omega - \Gamma)$ and the jump $[u]$ exists along Γ, or (b) $u \in C^1(\Omega - \Gamma) \cap C(\Omega)$ and both $[u_x]$ and $[u_y]$ exist along Γ. Prove that u is a weak solution of the linear equation, i.e. for all C^∞ function $\phi(x,y)$ in Ω with compact support, we have

$$\int_\Omega [u(a\phi)_x + u(b\phi)_y + c\phi]dxdy = 0,$$

iff Γ is a charateristic curve of the equation.

B.

Find the field of the Monge cones for the equation $u^2(u_x^2 + u_y^2 + 1) = 1$. Then solve the Cauchy problem with the Cauchy data $u = 1/2$ on $y = x$. If the caustic curve exists for this solution, find it explicitly.

C.

Let $\Omega = \{(x,y) \mid 0 < y < 1, 0 < x < \infty\}$

(1)

Prove that the boundary value problem (BVP)
$\Delta u = 0$ on $\Omega; u(x,0) = u(x,1) = 0$ for $0 < x < \infty; u(x,y)$ is bounded in Ω.

has a unique solution if $u(0,y) = f(y)$ is a given continuous function in

$0 \le y \le 1$. Find this solution explicitly.

(2)

Must the BVP have a unique solution if $u_x(0,y) = f(y)$ is a given continuous function in $0 \le y \le 1$? Find all it's solution if they exist.

D.

Let $\Omega \subset R^n$ be a bounded domain with smooth boundary $\partial\Omega$. Conseder the Dirichlet problem for $u \in C^2(\Omega)$

$$\Delta u = 0 \text{ on } \Omega; u(x) = f(x) \text{ given on } \partial\Omega.$$

Suppose that $f : \partial\Omega \to R$ is continuous except at $p_0 \in \partial\Omega$ and $u \in C(\bar{\Omega} - \{p_0\})$ .Show that, if u is bounded, then this solution of the Dirichlet problem is the unique solution. However, if u is possible unbounded, then the Dirichlet problem may have more than one solution.

E.

Consider the Cauchy problem for the heat equation

$$u_t = \Delta u + f(x) \text{ for } x \in R^n \text{ , } t > 0; u(x,0) = 0 \text{ for } x \in R^n$$

$u(x,t)$ is bounded on for any fixed $a > 0$.

(1)

For a fixed point $y \in R^n$, find the solution $u(x,t)$ if $f(x) = \delta(x-y)$. Show that $lim_{t \to \infty} u(x,t)$ is the heat kernel at y.

(2)

Find condition on $f(x)$ such that the unique solution $u(x,t)$ has the limit

$$t \infty u(x,t) = U(x), \text{ for each } x \in R^n$$

with $U \in C^2(R^n)$. Moreover, show that $U(x)$ must satisfy $\Delta U + f(x) = 0$ on R^n and $lim_{|x| \to \infty} U(x) = 0$.

F.

Solve the following wave equation

$$u_{tt} = \Delta u - m^2 u \text{ for } x \in R^3 \text{ , } t > 0; u(x,0) = f(x), u_t(x,0) = g(x) \text{ for } x \in R^3$$

where $m > 0$ is a constant. If $f(x)$ and $g(x)$ are of compact support, what is the domain of dependence of $u(x,t)$? Do we have the phenomena of the Huygen's principle for this equation?