臺灣大學數學系

八十七學年度博士班入學考試題

偏微分方程式

Choose 4 problems from below. 25 points each.

1. Suppose $u \in C^1[a,b]$ and $u(a) = u(b) = 0$. Prove the Poincare inequality

$$\int_a^b u^2\,dx \le C \int_a^b u_x^2\,dx$$

   for some constant C.
2. Consider the initial value problem for the following symmetric hyperbolic system

$$A_0(x,t)u_t + A(x,t)u_x + B(x,t)u = c(x,t), u \in R^n,$$
$$u(x,0) = f(x)$$

   where A_0, A are $n \times n$ symmetric matrices, $A_0 > 0$, and A_0, A, B, c, f are bounded smooth functions. Prove the existence and uniqueness theorems for this initial value problem by using the energy method.
3. Let $n = 2$ and Ω be the halfplane $x_2 > 0$.
   1. Derive formally the Poisson formula

$$u(x_1,x_2) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{x_2 f(y_1)}{(x_1-y_1)^2 + x_2^2} dy_1$$

      and show that this formula actually represents a bounded solution of the Dirichlet problem

$$\triangle u = 0 \text{ in } \Omega,\ u = f \text{ on } \partial\Omega,$$

      if f is bounded and continuous.
   2. Show that the maximal principle is satisfied by this solution.
4. Find the solution $u(x,t)$ of $u_{tt} - c^2 u_{xx} = 0$ in $x > 0, t > 0$, for which

$$u = f(x),\ u_t = g(x) \text{ for } t = 0, x > 0$$
$$u_t = \alpha u_x \text{ for } x = 0, t > 0,$$

   where α is a constant, $\alpha \neq c$, $f, g \in C^2$ and vanish near $x = 0$. Show that generally no solution exists when $\alpha = -c$.
5. Solve the following PDEs. You may express the solution in closed form, in implicit form, or in series form.

$$\begin{aligned} \Delta u &= 12(x^2 - y^2), \quad 0 < a^2 < x^2 + y^2 < b^2, \\ u(x, y) &= 0 \qquad \text{on } x^2 + y^2 = a^2, \\ \frac{\partial u}{\partial n}(x, y) &= 0 \qquad \text{on } x^2 + y^2 = b^2, \end{aligned}$$

1.

where n is the unit exterior normal of $x^2 + y^2 = b^2$, $0 < a < b$ are two constants.

$$\begin{aligned} u_t + uu_x + 2xu &= 0, \quad \text{for } t > 0,\ 0 < x < 1, \\ u(x, 0) &= 4 - x^2 \qquad \text{for } 0 < x < 1, \\ u(0, t) &= 1 \qquad \text{for } t > 0. \end{aligned}$$

2.

6. Let $\Delta u(x) = 0$ and $|u(x)| \leq M$ for $|x - \xi| < a$. Show that

$$|D^\alpha u(\xi)| \leq \left(\frac{m}{a}\gamma_n\right)^m M \text{ for } |\alpha| = m,$$

where

$$\gamma_n = \frac{2n\omega_{n-1}}{(n-1)\omega_n}$$

and ω_n is the volume of n-sphere.