臺灣大學數學系

八十六學年度博士班入學考試題

偏微分方程式

A.

Consider the following initial-boundary value problem:

$$\begin{cases} u_t = u_{xx} + 2u_x + 17u, & \text{for } \ 0 < x < 1, t > 0, \\ u(x,0) = f(x), & \text{in } \ 0 < x < 1, \\ u(0,t) = u(1,t) = 0, & \text{for } \ t > 0, \end{cases}$$

where $f(x)$ is continuously differentiable in $[0,1]$, and $f(0) = f(1) = 0$.

(a)

Define $\phi(t) = \int_0^1 |u(x,t)|^2 dx$. Prove that $\phi(t) \le e^{34t} \int_0^1 |f(x)|^2 dx$. [5 points]

(b)

Write down the general solution $u(x,t)$ in series form. Show that the series representing $u(x,t)$ is indeed a classical solution of the problem. Hence conclude that this problem is a well-posed problem. [12 points]

(c)

For $f(x) = e^{-x} \sin \pi x$, find $u(x,t)$. Show that $u(x,t)$ takes its maximum value in the interior of $0 < x < 1$, $t > 0$. Why does this fact not contradict with the maximum principle for parabolic equations? [4 points]

(d)

Characterize those $f(x)$ such that $\lim_{t \to \infty} u(x,t)$ exists, and is a steady solution of the equation. What are the steady solutions? [4 points]

B.

Consider the exterior Dirichlet problem for harmonic functions in the (x,y) plane:

$$\begin{cases} \Delta u(x,y) = 0, & \text{for } \ x^2 + y^2 > R^2, \\ u(x,y) = \varphi(x,y) & \text{on } \ x^2 + y^2 = R^2, \\ \lim\limits_{x^2+y^2 \to \infty} u(x,y) = 0, & \end{cases}$$

where $\varphi(x,y)$ is continuous on the circle $x^2+y^2=R^2$, and $R>0$ is a constant.

(a)
Prove that the problem has at most one solution. [6 points]

(b)
Is the problem always solvable? If yes, prove it. If no, find the necessary and sufficient conditions of $\varphi(x,y)$ so that the problem is solvable. [6 points]

(c)
If the problem is solvable, write down the explicit form (either in series or integral form) of $u(x,y)$. [13 points]

C.
Find $u(x,t)$ on the half plane $x \in R$, $t>0$ such that

$$\begin{cases} u_{tt} = \begin{cases} 4u_{xx}, & \text{for } x<0, t>0, \\ u_{xx}, & \text{for } x>0, t>0, \end{cases} \\ u(x,0) = u_t(x,0) = 0, \quad \text{for } x \in R, \\ u(0+,t) = u(0-,t), \quad \text{for } t>0, \\ u_x(0+,t) - u_x(0-,t) = a \sin wt, \text{ for } t>0, \end{cases}$$

where a and w are two constants. Is $u(x,t)$ twice differentiable at every point of $x \neq 0$, $t>0$? If yes, prove it. If no, find the set of all discontinuity points of $u_x(x,t)$ on $x \neq 0$, $t>0$. Is this set consisting of characteristic curves? [25 points]

D.
Consider the Cauchy problem for the following first-order equation:

$$\begin{cases} yu_x - xu_y = 0, & \text{in } (x,y) \in R^2, \\ u(\cos\theta, \sin\theta) = g(\theta), & \text{for } -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}, \\ u(0,y) = f(y), & \text{for } -1 \le y \le 1, \end{cases}$$

where $g(\theta)$ and $f(y)$ are two smooth (i.e. many times differentiable) functions, and $g(\frac{\pi}{2}) = f(1)$, $g(-\frac{\pi}{2}) = f(-1)$.

(a)
Find all the points on the Cauchy data which are characteristic with respect to the problem. [5 points]

(b)
Is this Cauchy problem always solvable? If yes, prove it. If no, find the necessary and sufficient conditions on $g(\theta)$ and $f(y)$ such that a local solution to this Cauchy problem exists. [7 points]

(c)

Under the conditions of (b), is the solution always unique? If yes, find the maximal domain of existence of $u(x, y)$. [7 points]

(d)

Is this Cauchy problem well-posed? Prove your answer. [6 points]

[Note: By a solution of this Cauchy problem, we always mean classical differentiable solution.]