台灣大學數學系

九十三學年度博士班入學考試題

幾何與拓樸

June 4, 2004

(1)

(25 pts) i) Let $X=\sum_{i=1,2} a_i(x_1,x_2)\frac{\partial}{\partial x_i}$ and $Y=\sum_{i=1,2} b_i(x_1,x_2)\frac{\partial}{\partial x_i}$ be vector fields (on $\mathbf{R}^2$). Find a formula for the bracket $[X,Y]$. ii) Let ω be a differential p-form on an n-dimensional differentiable manifold M, and d be the exterior derivative on M. Find the formula for $d\omega$ by using local coordinates, and show that your answer is independent of the choice of coordinates. pt

(2)

(25 pts) Let M be a simply connected, n-dimensional differentiable manifold. Let ω be a differential 1-form. Suppose that ω is closed, i.e. $d\omega=0$. Show that ω is exact, i.e. there exists a differentiable function f on M such that $df=\omega$. pt

(3)

(25 pts) Let T be a topological torus, i.e. diffeomorphic to $\mathbf{R}^2/\mathbf{Z}^2$. Let g be a Riemannian metric on T, written as $g=a(x,y)dx^2+2b(x,y)dxdy+c(x,y)dy^2$ where (x,y) are the Euclidean coordinates on $\mathbf{R}^2$. Suppose the Gaussian curvature $K(g)\leq 0$ everywhere. Find all possible solutions of g. pt

(4)

(25 pts) Prove or disprove that the tangent bundle $T(S^2)$ of the 2-dimensional sphere S^2 is a topologically non-trivial vector bundle, i.e. $T(S^2)$ is not equivalent to the trivial bundle $\mathbf{R}^2\times S^2$ over S^2.