國立臺灣大學數學系
九十七學年度博士班入學考試試題
科目：離散數學

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(20%) **1.** Recall that a family $\mathcal{F}$ of subset of a set $X$ is intersecting if $A \cap B \neq \emptyset$ whenever $A, B \in \mathcal{F}$. A family $\mathcal{F}$ of subsets of $X$ is called regular if every element in $X$ lies in a constant number $r$ of elements of $\mathcal{F}$. We use $[n]$ to denote $\{1, 2, \ldots, n\}$ .

(a) Prove that an intersecting family $\mathcal{F}$ of subsets of $[n]$ satisfies $|\mathcal{F}| \leq 2^{n-1}$.

(b) If $n$ is not a power of 2, construct a regular intersecting family of subsets of $[n]$, having size $2^{n-1}$.

(c) if $n = 2, 4$ or $8$, show that there is no such family.

(20%) **2.** Let $A$ and $B$ be two $m$ by $n$ matrices with entries in $\{0, 1\}$. An exchange operation substitutes a sub-matrix of the form $\binom{0\ 1}{1\ 0}$ for a sub-matrix of the form $\binom{1\ 0}{0\ 1}$ or vice versa. Prove that if $A$ and $B$ have the same list of row sums and have the same list of column sums, then $A$ can be transformed into $B$ by a sequence of exchange operations.

(20%) **3.** Recall that a doubly stochastic matrix is a non-negative real matrix in which every row and every column sums to 1. Let $P$ be the set of all $n$ by $n$ doubly stochastic matrices. Prove that $P$ is a polytope, and determine all its extremal points.

(20%) **4.** Prove that every $n$-vertex simple graph $G$ with no $r+1$-clique has at most $(r-1)n^2/(2r)$ edges. Use this fact to prove that if $G$ has $m$ edges then $\omega(G) \geq \lceil n^2/(n^2 - 2m) \rceil$.

(20%) **5.** Suppose $R(p, q)$ denotes the standard Ramsey number. Prove that for $p \geq 2$ and $q \geq 2$ we have $R(p, q) \leq R(p-1, q) + R(p, q-1)$. If $R(p-1, q)$ and $R(p, q-1)$ are both even, prove that $R(p, q) \leq R(p-1, q) + R(p, q-1) - 1$.