臺灣大學數學系

九十學年度博士班入學考試題

分析

1 .

(a)

Show that the Cantor Set is a nowhere dense perfect subset of $[0,1]$.

(b)

Let $f:[0,1]\to[0,1]$ be the Cantor-Lebesgue function.Find the length of the curve $y=f(x)$, $0\le x\le 1$(By defnition).

2.

Let f be a real-valued differentiable function on (a,b)

(a)

Does f' have to be Lebesgue measurable ? Justify your answer.

(b)

If f' is of bounded variation.Show that f' is continuous on (a,b)

3.

Let f_n, $f\in L^2(R)$ for all n.

(a)

If $f_n\to f$ a.e. and $\|f_n\|_2\le M\forall n$, show that $f_n\to f$ weakly in $L^2(R)$.

(b)

If $f_n\to f$ weakly in $L^2(R)$,show that there is a constant M such that

$\|f_n\|_2\le M\forall n=1,2\ldots$

4.

Show that

(a)

$C[0,1]$ is separable (i.e. $\exists$ countable dense subset).

(b)

$L^\infty(0,1)$ is not separable.

5.

(a)

Let $f \in L^1[0.1]$ and $F(x) = \int_0^1 f(t)\sin(xt)dt$, for all real x.Show that $F(x)$ is differentiable for each x.

(b)

Let $Y = C[0,1]$ with supremum norm and $X = C^1[0,1]$ (a subset of Y),is the mapping $f(x) \to f'(x)$ from X into Y continuous?Justify your answer.

6.

We call f a unit function if f is analytic in $U = (z : |z| < 1)$,continuous in $D = (z : |z| \leq 1)$, $|f(z)| \leq 1$ in D. and $|f(z)| = 1$ when $|z| = 1$.Prove the following statement:

(a)

If f is a unit function and has no zeros in U, then f is a constant.

(b)

If f is a non-constant function,then there exist $z_1, z_2, \ldots, z_n$ in U and a real constant k such that

$$f(z) = \exp(ik) \prod_{j=1}^{n} \frac{z_j - z}{1 - \bar{z}_j z}$$