臺灣大學數學系

八十六學年度博士班入學考試題

分析

1. **a.** find a function which is Lebesgue integrable but not Riemann integrable.
   b. Find a real-valued function f on $[0,1]$ such that f' is Lebesgue integrable and the following fundamental theorem of calculus is not valid, i.e.
   $$f(1) - f(0) \neq \int_{[}^{1} f'(x)dx.$$

2. Let f, f_k, $k = 1, 2, 3, \cdots, \in L^p([0,1])$ with $1 \leq p \leq \infty$. Show that if $f_k \longrightarrow f$ a.e. and $||f_k||_{L^p} \longrightarrow ||f||_{L^p}$, then $||f - f_k||_{L^p} \longrightarrow 0$.

3. If $A \subset [0, 2\pi]$ is Lebesgue measurable, prove that
   $$\lim_{n\to\infty} \int_A \sin nx dx = \lim_{n\to\infty} \int_A \cos nx dx = 0.$$

4. Let μ be the Lebesgue measure on $[0,1]$, $f \geq 0$ be measurable and $\lambda(y) = \mu(\{x \in [0,1] : f(x) > y\})$. Prove that
   $$\int_0^1 f^p(x)dx = p\int_0^\infty y^{p-1}\lambda(y)dy$$
   if $1 \leq p \leq \infty$.

5. Show that as $n \longrightarrow \infty$, $f_n(z) = \prod_{k=1}^{n}(1 + \frac{1}{k^z})$ converges uniformly on any compact subset of the half plane $\{z \in C : \text{Re } z > 1\}$ to a holomorphic function.