臺灣大學數學系

九十二學年度博士班入學考試題

代數

1

Show that $n^{13} - n$ is divisible by 2730 for any integer n.

2

Let T be a linear transformation on a finite dimensional vector space over k. Show that for any irreducible polynomial $f(t) \in k[t]$, if $f(T)$ is not onto, then $f(t)$ divides the characteristic polynomial of T.

3

Let A be an $n \times n$ matrix over a field k with $n > 1$, and adj A be the adjoint of A.

(a)

If A is invertible, prove that $\mathrm{adj}(\mathrm{adj}(A)) = (\det(A))^{n-2}A$.

(b)

Does the statement in (a) hold for singular matrix A?

4

Let p be a prime and $(\mathbb{Z}/(p^n))^\times$ be the multiplicative group of the ring $\mathbb{Z}/(p^n)$. That is, $(\mathbb{Z}/(p^n))^\times = \{a : 1 \leq a \leq p^n \text{ and } a \text{ is relative prime to } p\}$.

(a)

Let p be an odd prime and $n \geq 2$. Show that $1+p$ is a generator of the cyclic Sylow-p group of $(\mathbb{Z}/(p^n))^\times$.

(b)

Determine the group structures of $(\mathbb{Z}/(8))^\times$ and $(\mathbb{Z}/(16))^\times$?

5

Let $k[[x]] = \{a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots + a_nx_n + \cdots : a_i \in k\}$ be the ring of formal power series over the field k.

(a)

Find all units (invertible elements) in $k[[x]]$.

(b)

Classify all ideals in $k[[x]]$.

(c)

Classify all maximal ideals in $k[[x]]$.

(d)

Classify all prime ideals in $k[[x]]$.

6

Let F be a field , and K a finite extension of F. Let a be algebraic over K. Show that

(a)

$$[K(a):K] \leq [F(a):F].$$

(b)

$$[K(a):F(a)] \leq [K:F].$$