臺灣大學數學系

八十九學年度博士班入學考試題

代數

1.

(20%) Let G be a finite abelian group and $x, y \in G$. Let the order of x be m and the order of y be n.

(1)

Prove or disprove:

(i)

If $(m, n) = 1$, then the order of xy is mn.

(ii)

The order of xy is $[m, n]$ (L.C.M. of m and n).

(2)

Show that G contains a cyclic subgroup of order $[m, n]$.

2.

(20%) Find all homomorphisms ϕ for the given groups:

(1)

$\phi : \mathbf{Z}_6 \times \mathbf{Z} \to \mathbf{Z} \times \mathbf{Z}_{10}$. ($\mathbf{Z}_n$ is the cyclic group of order n.)

(2)

$\phi : \mathbf{S}_4 \to \mathbf{S}_3$. ($\mathbf{S}_n$ is the symmetric group of degree n.)

3.

(15%) Let A be any complex $n \times n$ matrix. Show that $I + A^* A$ is nonsingular. ($A^* = \bar{A}^t$ where $\bar{\ }$ denotes the complex conjugation and ^{t} denotes the transpose.)

4.

(15%) Let R be a commutative ring in which every element x satisfying $x^2 = x$.

(1)

Show that $2x = 0$ for all $x \in R$.

(2)

Show that every finitely generated ideal in R is principal.

5.

(15%) Let R be a right artinian ring, $n \in N$. Show that the matrix ring $M_n(R)$ is right artinian.

6.

(15%) Let $E = F(t)$ where t is transcendental over the field F. Let $u = \frac{f(t)}{g(t)} \in E$, where $(f(t), g(t)) = 1$. Show that t is algebraic over $F(u)$ with degree

$\max\{\deg f, \deg g\}$.