臺灣大學應用數學科學研究所 109 學年度碩士班甄試試題
科目：機率統計

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1. (15%) Let $Y = \exp(Z)$ and Z be a normal random variable with mean μ and variance σ^2. Compute the mean and variance of Y.

2. (15%) Let X, Y, and Z be independent $N(0,1)$. Let Θ, Φ, and R be the corresponding random variables with $X = R\sin\Phi cos\Theta$, $Y = R\sin\Phi\sin\Theta$, and $Z = R\cos\Phi$. Find the joint distribution of (Θ, Φ, R).

3. (20%) Let $X_1, \ldots, X_n$ be a random sample from a normal distribution with mean μ and variance σ^2. Derive the distribution of $\frac{\sqrt{n}(\bar{X}_n-\mu)}{S_n}$, where $\bar{X}_n = \frac{1}{n}\sum_{i=1}^n X_i$ and $S_n^2 = \frac{1}{(n-1)}\sum_{i=1}^n (X_i - \bar{X}_n)^2$ are the sample mean and sample variance, respectively.

4. (10%) (15%) Let $X_1, \ldots, X_{n+1}$ be a random sample from $Bernoulli(\pi)$ and $h(\pi) = P(\sum_{i=1}^n X_i > X_{n+1}|\pi)$. Find the maximum likelihood estimator of $h(\pi)$ and derive its asymptotic distribution.

5. (10%) (15%) Let $\{x_1, \ldots, x_n\}$ be observed values of a random sample $\{X_1, \ldots, X_n\}$ from a normal distribution with mean μ and variance σ^2, where σ^2 is an unknown constant. Express the corresponding p-value and power function of the likelihood ratio test for the hypotheses $H_0 : \mu \geq \mu_0$ versus $H_A : \mu < \mu_0$.