臺灣大學應用數學科學研究所 103 學年度碩士班甄試試題

科目：機率統計

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1. Let $X_1, \ldots X_n$ be a random sample from $Uniform(\theta_1, \theta_2)$, where $\theta_1 < \theta_2$.

(1a) (10%) Calculate the correlation of $M = \max\{X_1, \ldots, X_n\}$ and $m = \min\{X_1, \ldots, X_n\}$.

(1b) (10%) Find the uniformly minimum variance unbiased estimator of $\theta_2 - \theta_1$.

2. (10%) Suppose that $\sqrt{n}(Y_n - \mu) \xrightarrow{d} N(0, \sigma^2)$ and $g(\cdot)$ is a function with $g^{(1)}(\mu) = 0$ and $g^{(2)}(\mu) > 0$ being continuous. Find an approximated probability of $P(g(Y_n) \leq x)$.

3. Let N_t denote the number of events occurring within the time period $[0, t]$ and T be the time between two successive events.

(3a) (8%) Write the necessary conditions so that N_t follows a Poisson distribution with rate λ.

(3b) (7%) Derive the density function of T.

4. Let $X_{11}, \ldots, X_{1n_1}, \ldots, X_{k1}, \ldots, X_{kn_k}$ be k, $k > 2$, independent random samples from $N(\mu_1, \sigma^2), \ldots, N(\mu_k, \sigma^2)$, respectively.

(4a) (5%) (5%) Write the likelihood function for $\{(X_{11}, \ldots, X_{1n_1}), \ldots, (X_{k1}, \ldots, X_{kn_k})\}$ and derive the maximum likelihood estimator of $(\mu_1, \ldots, \mu_k, \sigma^2)$.

(4b) (10%) Find the rejection region of the likelihood ratio test at level α, $0 < \alpha < 1$ for $H_0 : \mu_1 = \ldots = \mu_k$ versus $H_A : \mu_i \neq \mu_j$ for some $i \neq j$.

5. Let $X_{11}, \ldots, X_{1n_1}$ and $X_{21}, \ldots, X_{2n_2}$ be independent random samples from $F_1(x)$ and $F_2(x)$, respectively.

(5a) (8%) Write the test statistic of the Mann-Whitney test for $H_0 : F_1(x) = F_2(x)$ versus $H_A : F_1(x) \neq F_2(x)$ for some x.

(5b) (5%) (7%) Derive the mean and variance of the test statistic under H_0.

6. (15%) Let $X_1, \ldots, X_n$ be a random sample from a population with p.d.f. $f(x)$ and c.d.f. $F(x)$. Show that $\sqrt{n}(M_n - \theta)$ is asymptotically normal with mean zero and variance $1/(2f(\theta))^2$, where M_n and θ are separately the sample median and population median.