臺灣大學應用數學科學研究所 108 學年度碩士班甄試試題

科目：微分方程與線性代數

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1. (40%)

Let $V:\mathbb{R}^n \to \mathbb{R}$ be a smooth and non-constant function. Consider the initial-value problem

$$\begin{cases} E\begin{bmatrix} p' \\ q' \end{bmatrix} = \begin{bmatrix} \nabla V(p) \\ q \end{bmatrix}, \ t>0, \\ p(0)=p_0 \in \mathbb{R}^n, q(0)=q_0 \in \mathbb{R}^n, \end{cases} \tag{1}$$

where $E=\begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix} \in \mathbb{R}^{2n\times 2n}$ and $I_n = \mathrm{diag}[1,\cdots,1] \in \mathbb{R}^{n\times n}$ are constant matrices. Answer the following questions:

A. Calculate matrix E^2 (10%)

B. Find eigenvalues of matrix E (10%)

C. Find a function $H=H(p,q)$ such that $\frac{d}{dt}H(p,q)=0$ for $t>0$ and (p,q) is a solution of system (1). Justify your answer. (10%)

D. Find a function $V=V(p)$ such that $(0,0)$ is a stable equilibrium of system (1). Justify your answer. (10%)

2. (40%)

Let $A=(a_{ij}) \in \mathbb{R}^{n\times n}$ and $a_{ij} = \begin{cases} -1 & \text{if } i=j \\ 1 & \text{if } j=i+1 \\ 0 & \text{otherwise} \end{cases}$

A. Can matrix A be invertible? (10%)

B. Calculate $\lim_{k\to\infty} \frac{A^k}{k!}$ (10%)

C. Calculate e^A (10%)

D. Prove that the zero vector is an asymptotically stable equilibrium of the system $x' = Ax$ (10%)

3. (20%)

Solve the initial value problem

$$y' = \frac{3x^2}{3y^2-4}, \quad y(1)=0$$

and determine the interval in which the solution is valid.