臺灣大學應用數學科學研究所 104 學年度碩士班甄試試題

科目：微分方程與線性代數

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1. (30%)

   Let $\phi : \mathbb{R}^n \to \mathbb{R}$ be a smooth and non-constant function. Assume that the third derivatives of ϕ are nonzero functions. Consider the initial-value problem

$$\begin{cases} \dfrac{dx(t)}{dt} = -\nabla\phi\big(x(t)\big), \ t > 0, \\ \quad x(0) = x_0 \in \mathbb{R}^n, \end{cases} \tag{1}$$

   and the solution is denoted as $x(t; x_0) \in \mathbb{R}^n$ for $t > 0$. Answer the following questions:

   A. Uniqueness of (1) holds true? Why? (5%)

   B. Suppose $\nabla\phi(y) = 0$ for some $y \in \mathbb{R}^n$.

      Is y an equilibrium of (1)? Why? (5%)

   C. Find a condition of ϕ such that y is asymptotically stable. Justify your answer. (10%)

   D. Suppose ϕ is strictly convex. Is it possible that (1) has a nontrivial periodic solution? Justify your answer. (10%)

2. (20%)

   Let A be the set of maps $f : \mathbb{R} \to \mathbb{R}$ which are solutions to the differential equation $f''' + f'' - 2f = 0$. Prove that A is a vector space and find its dimensions.

3. (20%)

   Let E be the $n \times n$ matrix with all entries 1.

   A. Is E diagonalizable? (5%)

   B. Find the characteristic polynomial of E. (15%)

4. (20%)

   Let $A = \begin{pmatrix} 1 & 2 \\ 1 & -1 \end{pmatrix}$.

   Express A^{-1} as a polynomial in A with real coefficients.

5. (10%)

   Find real valued functions of a real variable, $x(t)$, $y(t)$, $z(t)$, such that

   $x' = y, \quad y' = z, \quad z' = y$

   and $x(0) = 1, \quad y(0) = 2, \quad z(0) = 3$.