臺灣大學應數所 102 學年度碩士班甄試試題

科目：微積分　　2012.10.19

Calculus

2012.10.20

(1) (15 pts) Find the least constant c such that

$$(|x_1| + |x_2| + \cdots + |x_{2011}|)^2 \le c(x_1^2 + x_2^2 + \cdots + x_{2011}^2)$$

for all real values of $x_1, x_2, \ldots, x_{2011}$. (For emphasis, let us repeat that you are asked to find the least such c, not just some such c.)

(2) (15 pts) Let $f : [0,1] \to R$ be continuous. Determine $\lim_{n\to\infty} \int_0^1 nx^n f(x)dx$. Prove your answer.

(3) (20 pts) Let C be the oriented curve parameterized by $\vec{r}(t) = (\cos t, \sin t, 8 - \cos^2 t - \sin t)$, $0 \le t < 2\pi$, and let $\vec{F}$ be the vector field $\vec{F}(x,y,z) = (z^2 - y^2, -2xy^2, e^{\sqrt{z}} \cos z)$. Evaluate $\int_C \vec{F} \cdot d\vec{r}$.

(4) (30 pts) Let $\lambda_1 < \lambda_2 < \lambda_3 < \cdots$ be the consecutive strictly positive solutions of the equation $x = \tan x$.

(a) Give your estimates on those λ_n that you use and explain how you obtain it.

(b) Does the series $\sum_{n=1}^{\infty} \lambda_n^{-2}$ converge? Justify your answer.

(5) (20 pts) Let $\{a_n\}$ be a sequence of real numbers such that the series $\sum_{n=1}^{\infty} a_n$ converges. Prove that

$$\frac{1}{n}\sum_{k=1}^{n} ka_k \to 0$$

as $n \to \infty$. (Hint: For any sequence $\{b_n\}$ of real numbers, if $b_n \to b \in R$ as $n \to \infty$, then $\frac{1}{n}\sum_{k=1}^{n} b_k \to b$ as $n \to \infty$. You may use this fact without proof.)