# 臺灣大學數學系
# 九十七學年度碩士班甄試試題
## 科目：機率 2007.11.02

1. (8%) Suppose that on average only 2 out of 1000 people of certain population would have certain rare disease. A test for this disease is assumed to be correct 99% of the time: if a person has the disease, the test result is positive with probability 0.99, and if the person does not have the disease, the test result is negative with probability 0.99. Find the probability that a person has the disease when he/she is tested positive.

2. (16%) Holden gets calls from deceivers everyday. The probability that Holden is deceived into losing money on day $n$, given that he has not been deceived on any of the preceding days is $1-\left(\frac{n}{n+1}\right)^2$, $n \geq 1$. Let $T$ be the number of days until he is deceived (for the first time).

   (a) Find the probability mass function of $T : p_T(n) = \mathbf{P}(T = n)$.

   (b) Show that $\mathbf{P}(T < \infty) = 1$.

   (c) Evaluate $\mathbf{E}[T]$, the expectation of $T$. Give a closed form answer.

   (d) Evaluate $\mathbf{Var}(T)$, the variance of $T$.

3. (16%) Answer the following questions completely and explicitly.

   (a) State any version of *Law of Large Numbers*. No proof is needed.

   (b) State the *Central Limit Theorem*. No proof is needed.

   (c) Sketch an application of either theorem. No proof is needed.

4. (20%) Let $V$ and $U$ be two random variables on a probability space.

   (a) Suppose that the distribution function $F_V$ of $V$ is continuous and strictly increasing. Find the probability law of the random variable $Z = F_V(V)$.

   (b) Suppose that $U$ is uniform on $(0,1)$. Find a real valued function $h$ such that $h(U)$ is an exponential random variable with parameter $\lambda > 0$.

5. (20%) Suppose that the customer entering a post office is male with probability 2/5, and is female with probability 3/5, independently of earlier customers. Also suppose that the number of customers entering a post office within 10 minutes follows a Poisson distribution with parameter 15. Let $M$ and $W$ be the random number of males and females entering a post office within 10 minutes, respectively.

   (a) Find the joint probability mass function of $M$ and $W : p_{M,W}(n,k) = \mathbf{P}(M = n, W = k)$.

   (b) Are $M$ and $W$ independent? Give your reason.

   (c) Find the probability that there are more than 2 males and less than 3 females entering a post office within 10 minutes.

6. (20%) Let $\mathcal{U} = \{(x,y) : x^2 + y^2 \leq a^2, y \geq 0\} \subset \mathbb{R}^2, a > 0$, and a point $\mathbf{A} = (a_1, a_2)$ is picked *randomly* in $\mathcal{U}$ so that the probability that $\mathbf{A}$ is in $O \subset \mathcal{U}$ is proportional to the area of $O$, where $O$ is any "nice" (for example, open or closed) subset of $\mathcal{U}$. Let random variable $X = |a_1|$ be the distance from the $y$-axis of $\mathbf{A}$, and random variable $Y = a_2$ be the $y$-coordinate of $\mathbf{A}$.

   (a) Are $X$ and $Y$ independent? Give your reason.

   (b) Find the distribution functions $F_X(x), F_Y(y)$ of $X$ and $Y$, respectively.

   (c) Evaluate $\mathbf{E}[Y]$.