臺灣大學數學系
九十六學年度碩士班甄試試題
科目：線性代數

2006.11.3

LINEAR ALGEBRA

11/03 2006

You should include in your answer every piece of computations and every piece of reasonings so that corresponding partial credit could be gained.

(1) Find the eigen values and corresponding eigen vectors of the matrix

$$A = \begin{bmatrix} 5 & 12 \\ -2 & -5 \end{bmatrix}. \qquad \text{(15 points)}$$

And find a matrix B such that $B^{-1}AB$ is diagonal (5 points).

(2) Let $t = (t_1, t_2, t_3)$ where t_1, t_2, t_3 are real numbers. For a vector $a = (a_5, a_4, a_3, a_2, a_1, a_0) \in \mathbb{R}^6$, let

$$f_a(x) = a_5x^5 + a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0,$$

and define

$$\Phi_t(a) = (f_a(t_1), f_a'(t_1), f_a(t_2), f_a'(t_2), f_a(t_3), f_a'(t_3)) \in \mathbb{R}^6,$$

here f_a' is the derivative of the polynomial function f_a. And let A_t be the matrix corresponding to the linear transformation Φ_t.

(a) Let $t = (-1, 0, 1)$. Find the matrix A_t, show that A_t is invertible and find a such that $\Phi_t(a) = (1, 1, 0, 3, 0, 6)$. (20 points)

(b) In general, for a given t such that t_1, t_2, t_3 are *distinct*, is the matrix A_t always invertible? Prove or disprove it. (15 points)

(3) An *invertible* linear transformation $T : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is called *grata* if it satisfies

$$(T(v), v) = 0, \forall v \in \mathbb{R}^n,$$

here (v, u) denotes the standard inner product for vectors u, v in the Euclidean space $\mathbb{R}^n$.

(a) Let $n = 2$. Show that every grata linear transformation must be of the form $T = \lambda R$ where $\lambda \in \mathbb{R}$, $\lambda \neq 0$ and R is the rotation of the plane $\mathbb{R}^2$ about the origin counterclockwise by the 90^o angle. (15 points)

(b) Show that for $n = 3$ there is no grata linear transformation. (15 points)

(4) Suppose A is a 10×10 square matrix satisfying $A^{100} = 0$. Is it necessary that $A^{99} = 0$? Prove or disprove it (15 points)