Linear Algebra

1. Are the following statements true or false? If true, give a proof. If false, give a counterexample.

a. If V and W are vector spaces, then $V \cap W$ is a vector space.
b. If V and W are vector spaces, then $V \cup W$ is a vector space.
c. The only $n \times n$ matrix that is both diagonalizable and nilpotent is the zero matrix.
d. If A and B are both nilpotent $n \times n$ matrices, then AB is a nilpotent $n \times n$ matrix.

2. Let V be a finite-dimensional vector space over $\mathbb{C}$. Let $T : V \to V$ be a linear map. Suppose that $W \subseteq V$ is a T-invariant subspace, i.e. $T(W) \subseteq W$.

3. Let A be an invertible $n \times n$ matrix and let N be a nilpotent $n \times n$ matrix. Suppose that $AN = NA$. Prove that $A - N$ is invertible.

4. Let V be a finite-dimensional real vector space equipped with an inner product $\langle \cdot, \cdot \rangle$. Let $v_1, \cdots, v_n$ be a set of non-zero vectors in V such that $\langle v_i, v_j \rangle \leq 0$, for all $i \neq j$.

a. Suppose that $v_1, \cdots, v_n$ are linearly dependent. Prove that there exists a non-trivial linear combination $\sum_{i=1}^{n} \lambda_i v_i = 0$, with $\lambda_i \geq 0$, for all i.
b. Suppose there exists a linear map $f : V \to \mathbb{R}$ such that $f(v_i) > 0$, for all i. Prove that $v_1, \cdots, v_n$ are linearly independent.

5. Let V be a finite-dimensional complex vector space equipped with a Hermitian product $\langle \cdot, \cdot \rangle$. Let $d : V \to V$ be a linear map satisfying $d^2 = 0$, and let $\delta : V \to V$ be the adjoint map of d with respect to $\langle \cdot, \cdot \rangle$.

a. Prove that $d\delta x = 0$ implies that $\delta x = 0$, and $\delta d x = 0$ implies that $dx = 0$, for all $x \in V$.
b. Let $\Delta = d\delta + \delta d$. Prove that $\mathrm{Ker}\Delta = \mathrm{Ker}\delta \cap \mathrm{Ker}d$.
c. Prove that $\mathrm{Ker}\Delta \cap \Big(\mathrm{Im}\delta + \mathrm{Im}d\Big) = 0$.
d. Prove that $V = \mathrm{Ker}\Delta \oplus \Big(\mathrm{Im}\delta + \mathrm{Im}d\Big)$.
e. Prove that $\mathrm{Ker}d/\mathrm{Im}d \cong \mathrm{Ker}\Delta$.