臺灣大學數學系
九十一學年度第一學期碩士班甄試入學試題
線性代數甲
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1.

Let a $2n \times 2n$ matrix be given in the form $M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ where each block is an $n \times n$ matrix. Suppose that A is invertible and that $AC = CA$. Show that

$$Det \begin{bmatrix} A & B \\ C & D \end{bmatrix} = Det(AD - CB).$$

2.

Let $T : V \to V$ be a linear transformation on a finite dimensional vector space V.
(a) Show that there exists k such that Im$T^k =$ImT^m and Ker$T^k =$KerT^m for all $m \geq k$.

(b) Show that there exists n such that Ker$T^n \cap$Im$T^n = (0)$.

3.

Let $n \geq 2$ and N be an $n \times n$ matrix over a field such that $N^n = 0$ but $N^{n-1} \neq 0$ .Prove that N has no square root A (i.e. $A^2 = N$).

4.

Let T be a unitary linear transformation on a finite dimensional vector space V over $\mathbf{C}$ (i.e. $\langle Tx, Ty \rangle = \langle x, y \rangle$ for all $x, y \in V$). Prove that T has a unitary square root.