# 臺灣大學數學系

# 九十學年度碩士班甄試入學考試試題

# 線性代數

1.

Let $A$ be an invertible matrix over a field $F$.

a.

Suppose $A$ is diagonalizable. Is $A^{-1}$ diagonalizable? What are the eigenvalues of $A^{-1}$?

b.

Show that $A^{-1} = p(A)$, where $p(x)$ is a polynomial in $x$ with coefficients in $F$.

2.

Let $A$ be a complex $n \times n$ matrix. Prove that there exist complex matrices $A_s$ and $A_n$ with the following properties:

a.

$A_s$ is diagonalizable and $A_n$ is nilpotent.

b.

$A_s A_n = A_n A_s$ and $A = A_s + A_n$.

3.

Let $A$ be a real $n \times n$ matrix such that $A^t = -A$. Prove that every eigenvalue of $A$ is purely imaginary, i.e. if λ is an eigenvalue, then $\lambda = i\alpha, \alpha \in R$ and $i^2 = -1$.

4.

Let $V$ be a finite-dimensional vector space over $C$. Let $T : V \to V$ be a nilpotent linear map. Prove that for every $i \in N$ we have

$$\dim(\ker T^{i+1}) - \dim(\ker T^i) \geq \dim(\ker T^{i+2}) - \dim(\ker T^{i+1}).$$

5.

Let $V$ be a real vector space.

a.

Suppose that $J : V \to V$ a linear map such that $J^2 = -I$, where $I$ is the identity map. Prove that $\dim V$ is an even integer.

b.

Let $J_0$ be a $2n \times 2n$ matrix $\begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}$, where $I_n$ is the $n \times n$ identity matrix. Let $A$ be a $2n \times 2n$ matrix such that $A^t J_0 A = J_0$. Prove that $\lambda$ is an eigenvalue of $A$ if and only if $\frac{1}{\lambda}$ is an eigenvalue of $A$.