臺灣大學數學系112學年度碩士班甄試試題

科目：線性代數

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Notation: $\mathbb{R}$ is the set of real numbers and $\mathbb{C}$ is the set of complex numbers. If $F = \mathbb{R}$ or $\mathbb{C}$ and n is a positive integer, we denote by $\mathrm{M}_n(F)$ the set of $n \times n$ matrices with entries in F and by I_n the identity matrix in $\mathrm{M}_n(F)$.

Problem 1 (15 pts)**.** Let

$$\mathbf{v}_1 = (1, 2, 0, 4),\ \mathbf{v}_2 = (-1, 1, 3, -3),\ \mathbf{v}_3 = (0, 1, -5, -2),\ \mathbf{v}_4 = (-1, -9, -1, -4)$$

be vectors in $\mathbb{R}^4$. Let W_1 be the subspace spanned by $\mathbf{v}_1$ and $\mathbf{v}_2$ and let W_2 be the subspace spanned by $\mathbf{v}_3$ and $\mathbf{v}_4$. Find the dimension and a basis of $W_1 \cap W_2$.

Problem 2. Let

$$A = \begin{pmatrix} 0 & 1 & -1 \\ 3 & -4 & 1 \\ 3 & -8 & 5 \end{pmatrix}.$$

(1) (10pts) Find an invertible matrix $Q \in \mathrm{M}_3(\mathbb{C})$ such that

$$Q^{-1}AQ = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 12 \\ 0 & 1 & 1 \end{pmatrix}.$$

(2) (15pts) Find an invertible matrix $P \in \mathrm{M}_3(\mathbb{C})$ such that $P^{-1}AP$ is a diagonal matrix.

Problem 3. For any $A \in \mathrm{M}_2(\mathbb{C})$, define

$$\sin A = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} A^{2n+1}.$$

(1) (5pts) Evaluate $\sin \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}$.

(2) (15pts) Prove or disprove: There exists $A \in \mathrm{M}_2(\mathbb{R})$ such that

$$\sin A = \begin{pmatrix} 1 & 2022 \\ 0 & 1 \end{pmatrix}.$$

Problem 4 (20pts)**.** Let $A = (a_{ij}) \in \mathrm{M}_n(\mathbb{C})$ and let $\lambda_1, \lambda_2, \ldots, \lambda_n \in \mathbb{C}$ be roots of characteristic polynomial of A (counted with multiplicity). Show that

$$AA^* = A^*A \text{ if and only if } \sum_{1 \le i,j \le n} |a_{ij}|^2 = \sum_{k=1}^{n} |\lambda_k|^2.$$

Problem 5 (20pts)**.** Let $A, B \in \mathrm{M}_n(\mathbb{C})$. Suppose that all of the eigenvalues of A and B are positive real numbers. If $A^4 = B^4$, prove that $A = B$.