臺灣大學數學系 110 學年度碩士班甄試試題

科目：線性代數

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In the following, for a linear map $f : V \to V$, $\ker f$ and $\operatorname{im} f$ denote the kernel and the image of f, respectively.

1. Let V be a finite-dimensional complex inner product space. Let $d : V \to V$ be a linear map satisfying $d^2 = 0$. Let $\delta : V \to V$ be the adjoint of d and $\Delta = d\delta + \delta d$. Prove the following.

   (a) [5%] $d\delta x = 0$ implies that $\delta x = 0$, and $\delta d x = 0$ implies that $dx = 0$, for all $x \in V$.

   (b) [10%] $\ker \Delta = \ker d \cap \ker \delta$.

   (c) [10%] There is the orthogonal decomposition $V = \ker \Delta \oplus \operatorname{im} d \oplus \operatorname{im} \delta$.

   (d) [5%] There is the orthogonal decomposition $\ker d = \ker \Delta \oplus \operatorname{im} d$.

2. [10%] Let $V = \mathbb{R}^n$ be the space of column vectors, and M a positive definite symmetric $n \times n$ real matrix. Suppose the matrix $A \in M_n(\mathbb{R})$ satisfies $MAM^{-1} = A^t$. Show that there exists $P \in M_n(\mathbb{R})$ satisfying $P^t M P = I_n$ such that $P^{-1}AP$ is diagonal. (Here B^t denotes the transpose of the matrix B.)

3. (a) [10%] Let M be an invertible $n \times n$ complex matrix. Prove that there exists an invertible matrix A such that $A^2 = M$.

   (b) [10%] Let $n \geq 2$ and N be an $n \times n$ matrix over a field such that $N^n = 0$ but $N^{n-1} \neq 0$. Prove that there is no square matrix B such that $B^2 = N$.

4. [20%] Let V be a vector space over a field F and $u_1, \cdots, u_n \in V$ are linearly independent. Show that, for any $v_1, \cdots, v_n \in V$, $u_1 + \alpha v_1, \cdots, u_n + \alpha v_n$ are linearly independent for all but finitely many values of $\alpha \in F$.

5. [20%] Let P be an $n \times n$ matrix with coefficients in a field. Suppose $\operatorname{rank}(P) + \operatorname{rank}(I_n - P) = n$. Prove that $P^2 = P$.