臺灣大學數學系 108 學年度碩士班甄試試題

科目：線性代數

2018.10.19

1. Find all possible Jordan forms for 8×8 real matrices having $x^2(x-2)^3$ as minimal polynomial. (**20 points.**)

2. Let V be a vector space over a field $\mathbb{F}$ of infinite elements, and let $v_1, \ldots, v_n$ be vectors in V, where n is a positive integer. Suppose that $v_0 + zv_1 + \ldots + z^n v_n = 0$ for infinitely many z in $\mathbb{F}$. Prove that all v_i's are zero. (**20 points.**)

3. Let $V = M(n, \mathbb{R})$ be the vector space of all $n \times n$ matrices and $f : V \to \mathbb{R}$ be a linear transformation. Assume that $f(AB) = f(BA)$ for all $A, B \in V$ and $f(I_n) = n$, where I_n is the identity matrix in V. Prove that f is the trace function. (**20 points.** *Hint*: Consider the cases $A = E_{ij}$ and $B = E_{k\ell}$ for various E_{ij} and $E_{k\ell}$. Here E_{ij} denotes the matrix whose (i,j)-entry is 1 and whose other entries are 0.)

4. Let V be an n-dimensional vector space over $\mathbb{R}$ and $B : V \times V \to \mathbb{R}$ be a symmetric bilinear form on V. (*Symmetric* means $B(u,v) = B(v,u)$ for all $u, v \in V$. *Bilinear* means that B is linear in each of the two variables.)

(a) Let W be a vector subspace of V and let

$$W^{\perp} = \{u \in V : B(u,v) = 0 \text{ for all } v \in W\}.$$

Prove that if $\dim W = m$, then $\dim W^{\perp} \geq n - m$. (**10 points.** *Hint*: Choose a basis $\{v_1, \ldots, v_m\}$ for W and consider the map

$$u \longmapsto (B(u,v_1), \ldots, B(u,v_m))$$

from V into $\mathbb{R}^m$.)

(b) Prove that $V = W \oplus W^{\perp}$ if and only if the restriction of B to W is nondegenerate. (*Nondegenerate* means that $v = 0$ is the only vector of W such that $B(u,v) = 0$ for all $u \in W$.) (**15 points.**)

(c) Prove that if B is nondegenerate on V, then there is a nonnegative integer p with $p \leq n$ and a basis $\{v_1, \ldots, v_n\}$ such that

$$B(v_i, v_j) = \begin{cases} 1, & \text{if } 1 \leq i = j \leq p, \\ -1, & \text{if } p+1 \leq i = j \leq n, \\ 0, & \text{if } i \neq j. \end{cases}$$

(**15 points.**)